FACTORIZATION IN GROUP ALGEBRAS

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Let G be an arbitrary locally compact group, commutative or not. Then there exists on G an essentially unique left invariant Borel measure, the so-called Haar measure on G. With respect to this measure there exists the space of all integrable functions, denoted by $L^1(G)$. $L^1(G)$ froms a Banach algebra where the multiplication is defined by convolution, that is,

$$(f * g)(x) = \int_G f(t)g(t^{-1}x)\, dt.$$

The norm of an element is defined by $\|f\| = \int |f(t)|\, dt$. We then have the usual rule $\|f * g\| \leq \|f\| \cdot \|g\|$. In [2], [3] Rudin has considered the question of whether every function in $L^1(G)$ is the convolution of two other functions. For the case where G is the additive group of Euclidean n-space, or the n-dimensional torus, he has shown that this is the case. His methods do not extend to the case of arbitrary groups, because they depend upon the use of the Fourier transform, and particular functions in Euclidean n-space. In this paper we shall answer the question affirmatively for the general case.

As is well known, the algebra $L^1(G)$ possesses approximate identities in the sense of the following definition.

Definition. A Banach algebra B is said to have a left approximate identity, if there exists a constant C, such that given $\epsilon > 0$, and x_i, $1 \leq i \leq n$, elements of B, there exists $e \in B$, satisfying

$$\|e\| < C, \qquad \|ex_i - x_i\| < \epsilon. \tag{1}$$

Given elements x_i in $L^1(G)$, to find an element e as in the definition, it suffices to take a non-negative function, whose integral is 1, and whose support is contained in a sufficiently small neighborhood of the identity. Now we shall prove the following theorem.

Theorem 1. *If B is a Banach algebra with a left approximate identity, then for each element z of B, and $\delta > 0$, there exist elements x and y in B such that*

(a) $z = xy$

(b) *y belongs to the closed left ideal generated by z*

(c) $\|z - y\| < \delta$.

Proof. Let B^* be the algebra B with an identity formally adjoined. That is B^* consists of all elements of the form $x + \lambda I$ where I is the identity, and B^* is normed so that $\|x + \lambda I\| = \|x\| + |\lambda|$. Then B^* forms a Banach algebra. In the definition of an approximate identity we may assume that $C > 1$. Set $\gamma = 1/4C$. Then we have the following lemma.

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