REPRESENTATIONS OF EVEN FUNCTIONS (mod r), II. CAUCHY PRODUCTS

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1. **Introduction.** This paper has the same general purpose as a previous article [8] with a similar name (also to be denoted by I), namely, to study the representations of even functions (mod r) and, as a by-product of the theory developed, to obtain identities connecting various arithmetical functions. We recall that an arithmetical function $f(n, r)$ is even (mod r) if $f(n, r) = f((n, r), r)$ for all n.

It was evident in I that the representation theory of even functions (mod r) forms a fundamental source for certain types of arithmetical identities. This paper will further emphasize that fact by extending the scope of the method to cover new types of identities. The discussion will assume the definitions, formulas, and results stated or proved in I. The references listed in the bibliography of I will not be repeated here but will be indicated by means of a bracket symbol with the reference number starred. Regarding terminology and notation, the reader is referred to [8, §§1, 2].

The substance of the paper is concerned with the Cauchy product (mod r) of even functions $f(n, r)$, $g(n, r)$, defined by

$$h(n, r) = \sum_{n \equiv a+b \pmod{r}} f(a, r)g(b, r), \tag{1.1}$$

the summation extending over a, b (mod r) such that $n \equiv a + b \pmod{r}$. The law of multiplication (1.1) is the finite analogue of the classic Cauchy product. Regarding the theory of Cauchy multiplication and related modes of arithmetical composition, we mention work of Bervi [3], von Sterneck [13], von Schrutka [12], Bell [2], D. H. Lehmer [9], Carlitz [5], Rankin [11], and Uspensky [14].

At the foundation of this paper is the following orthogonality relation involving Ramanujan's sum $c(n, r)$: *Let d and e be divisors of r; then*

$$\sum_{n \equiv a+b \pmod{r}} c(a, d)c(b, e) = \begin{cases} rc(n, d) & \text{if } d = e, \\ 0 & \text{if } d \neq e. \end{cases} \tag{1.2}$$

This relation was proved by the author in [7, (3.10)] but is originally due, in an equivalent form, to Carmichael [6, §1], [6*, §1]. Still other forms of this relation exist, for example, [9*, (6)], but it is the particular form (1.2) that is fruitful for the applications of this paper.

We observe that the relation (1.2) implies [7, Theorem 3] that the even functions (mod r) with values in the complex field form a commutative semisimple subalgebra R of the ring of periodic functions (mod r), the ring operations being

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