COMPLETION OF SEMI-CONTINUOUS ORDERED COMMUTATIVE SEMIGROUPS

By A. H. Clifford

By an *ordered commutative semigroup* (abbreviated "o.c.s.") we shall mean a system $S(+, <)$ consisting of a set S endowed with a binary operation $+$ and a binary relation $<$ subject to the following axioms.

I. *S is a commutative semigroup with respect to* $+$, i.e. $+$ is associative, $a + (b + c) = (a + b) + c$ (all a, b, c in S), and commutative, $a + b = b + a$ (all a, b in S).

II. *S is a totally (= linearly = simply) ordered set with respect to* $<$.

III. (*Monotone condition.*) *If a and b are elements of S such that $a < b$, then $a + c \leq b + c$ for every element c of S.*

Note that we do not assume the strict monotone condition, $a < b$ implies $a + c < b + c$ for all c, which is equivalent to the conjunction of the monotone condition and the cancellation law, $a + c = b + c$ implies $a = b$. If an o.c.s. is a group, then it is an ordered Abelian group as customarily defined.

Let S be an o.c.s. The least upper bound (l.u.b.) of a subset A will, if it exists, be denoted by sup A, and the greatest lower bound (g.l.b.) by inf A. S is called (*conditionally*) *complete* if every subset of S, bounded from above, has a l.u.b. This is equivalent to the dual proposition. We shall omit the adverb "conditionally".

Let S and Σ be o.c.s.'s. By an *isomorphism* of S into Σ we mean a one-to-one mapping f of S into Σ such that $f(a + b) = f(a) + f(b)$ and $a < b$ implies $f(a) < f(b)$, for all a, b in S. We shall also say that f is an *embedding* of S in Σ, and shall often identify S with its image $f(S)$ in the usual way, and thus regard S as a subsemigroup of Σ. Σ will be called a *normal completion* of S if it is complete and if every element of Σ is the l.u.b. of some subset of S and also the g.l.b. of some subset of S. In this event, we shall also say that S is *normally embedded* in Σ. Two normal completions Σ and Σ' of S will be called *equivalent* if there is an isomorphism of Σ onto Σ' leaving each element of S fixed. We shall be concerned in this paper only with normal completions of a given o.c.s.

If S is an arbitrary (totally) ordered set, it possesses a normal completion which is unique to within equivalence, namely that constructed from Dedekind cuts in the well-known way. Consequently we shall find all possible normal completions of an o.c.s. S by taking its Dedekind completion Σ and extending the binary operation $+$ from S to all of Σ in every possible way. Suppose $\Sigma(+, <)$ and $\Sigma(\dot{+}, <)$ are two such ways. An equivalence mapping f of one of these onto the other preserves order and leaves the elements of S fixed. Since

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