ON THE GROUP OF THE COMPOSITION OF TWO GRAPHS

By Frank Harary

1. **Introduction.** We wish to construct a binary operation on linear graphs to be called composition, such that the group of the composition of two graphs is (in general) permutationally equivalent to the composition (the "Gruppenkranz" in Pólya [5]) of their groups. This problem was suggested by the work of Frucht [1] who gave the result for the case that the first graph in the composition is totally disconnected. We then compare the groups of three kinds of graph products, with the emphasis always on permutation groups rather than abstract groups.

In order to make this comparison, we review three different kinds of products of permutation groups. These are the direct sum (or direct product), the Cartesian product defined in [3], and the composition. After defining the composition of two graphs, we prove the theorem to the effect that the group of the composition of two graphs is the composition of their groups if and only if the graphs are not both complete.

We then turn to two other products of graphs: the (Cartesian) product studied by Shapiro [8] and Sabidussi [6], [7] and the join investigated in Zykov [10]. Conditions are given for the group of the product to be permutationally equivalent to the Cartesian product of the groups, and for the group of the join to be the direct sum of the groups.

2. **Three products of permutation groups.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be permutation groups of degree d_1 and d_2 and order n_1 and n_2 respectively. Let X_1 and X_2 be the (disjoint) object sets acted on by $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Let $\alpha_1 \in \mathfrak{A}_1$, $\alpha_2 \in \mathfrak{A}_2$ be written as products of disjoint cycles.

If $\mathfrak{A}$ and $\mathfrak{B}$ are permutation groups with object sets X and Y, we say that $\mathfrak{A}$ and $\mathfrak{B}$ are *permutationally equivalent* or *permutationally isomorphic* if there exist one-to-one mappings $f : X \to Y$ and $\theta : \mathfrak{A} \to \mathfrak{B}$ such that

$$f(\alpha x) = (\theta\alpha)f(x) \quad \text{for all} \quad x \in X, \alpha \in \mathfrak{A};$$

and θ is a group isomorphism: $\theta(\alpha_1\alpha_2) = (\theta\alpha_1)(\theta\alpha_2)$. We note that two permutation groups may be isomorphic or more precisely abstractly isomorphic without being permutationally isomorphic, e.g., the two cyclic groups of order 6 generated by the permutations (1 2 3 4 5 6) and (1 2 3) (4 5) respectively.

Direct Sum. The *direct sum* $\mathfrak{A}_1 + \mathfrak{A}_2$ has $X_1 \cup X_2$ as its object set so that its degree is $d_1 + d_2$. The order of $\mathfrak{A}_1 + \mathfrak{A}_2$ is $n_1 n_2$ since its elements are all the permutation products $\alpha_1\alpha_2$ obtained by juxtaposition of the permutations in $\mathfrak{A}_1$ with those in $\mathfrak{A}_2$.

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