# ON BANACH'S GENERALIZED LIMITS

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1. **Introduction.** As an application of the Hahn-Banach theorem, S. Banach [1; 33, 34] showed the existence of a 'generalized limit'; a translation-invariant positive functional of norm one on the space of all bounded real-valued functions on the line, or integers. Other proofs of the same fact depend on the compactness property of bounded sets in the conjugate space of a Banach Space. The proofs of Markov [9] and Kakutani [6; 1017–1019] invoke in particular the fixed-point property of the required functional. On the other hand, such fixed points are a standard feature of mean ergodic theorems [4], and this fact is used by M. M. Day [3; Theorem 2] who demonstrated the equivalence of one type of ergodicity of a semigroup in its bounded representations with the existence of an invariant mean.

The present work is concerned with two digressions on these researches. First, following [8] and [5], there is the question of the set of all Banach limits: what are their extreme values on a given function? For a typical general case, which happens to be ergodic, this question is answered in Theorem 3 via the exhibition of all these limits as the set of all cluster points of a certain net of functionals. The second digression is into the non-ergodic situation, but where mean-value operators of some weaker sort are available. In §§4 and 5 the operators $U_\rho$ do not lie in the set $K$ of Day's formulation [3, Definition 1], yet the net $\{U_\rho\}$ still has enough connection with the translation and rotation operators to produce Banach limits. The question of how many was treated in [11]; it is, for the case of the real line, further clarified in §4. §2 contains the fixed point theorem and construction needed throughout the sequel.

2. **The fixed point theorem.** We shall employ the following notations: $E$ is a locally convex linear topological space (real or complex), $E'$ its conjugate space, $L(E, E)$ the space of continuous linear mappings of $E$ into $E$. The value of $x' \varepsilon E'$ at $x \varepsilon E$ will be denoted $(x', x)$. $E'_\sigma$ is $E'$ in its weak topology (pointwise convergence on $E$). If $A$ is a subset of a linear space, $A_c$ denotes its convex extension, $A_L$ its linear extension, and $[A]$ its closed convex extension (where there is a topology). If $\{x_\alpha \mid \alpha \varepsilon A\}$ is a net [7; 65ff], $lp_\alpha x_\alpha$ will denote its set of cluster points; $lp_\alpha x_\alpha = \bigcap_{\alpha_0 \varepsilon A} (\overline{\bigcup_{\alpha > \alpha_0} x_\alpha})$ where the bar denotes closure. If $\mathfrak{J} \subset L(E, E)$, $\mathfrak{J}'$ will denote the set of elements in $L(E', E')$ adjoint to the members of $\mathfrak{J}$. Algebraic set notations, such as $\mathfrak{J}x$ for $\{Tx \mid T \varepsilon \mathfrak{J}\}$, and $\mathfrak{J}(A) = \{Tx \mid T \varepsilon \mathfrak{J}, x \varepsilon A\}$ will be used. The zero of $E$ will be denoted $\theta$, and $\theta'$ the zero of $E'$.

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