REMARK ABOUT WINTNER'S COMPARISON THEOREM

By Norman Levinson

Professor Wintner died before it was possible for him to add a few lines to clarify the applicability of the classical Sturm comparison theorem to the very interesting problem he solves in the preceding paper. This will be done here.

Let (a, b) be an open interval on which $r(t) > 0$ and let $r(a) = r(b) = 0$. Then it suffices to show that any real solution of $u'' + f(t)\, u = 0$ must have a zero on $(a, b]$ if $f(t) \geq g(t)$. To apply Sturm's theorem the behavior of $r(t)$ and $r'(t)$ as $t \to a + 0$ and as $t \to b - 0$ will be determined.

Since $r(a) = 0$, $x(a) = 0$ and hence

$$x(t) = x'(a)(t - a) - \int_a^t (t - \tau)F(\tau)x(\tau)\, d\tau \tag{1}$$

$$x'(t) = x'(a) - \int_a^t F(\tau)x(\tau)\, d\tau. \tag{2}$$

Because $F(t)$ is continuous $| x(t) |$ is bounded on $[a, b]$ and (1) shows that there is a constant K such that

$$| x(t) | \leq K(t - a) \qquad a \leq t \leq b. \tag{3}$$

Using (3) in (1) and (2)

$$x(t) = x'(a)(t - a) + O((t - a)^3)$$

$$x'(t) = x'(a) + O((t - a)^2).$$

Since $x(t) \not\equiv 0$, $\sqrt{x'(a) \cdot x'(a)} = c > 0$.
Hence $x(t) \cdot x(t) = c^2(t - a)^2 + O((t - a)^4)$
and so

$$r(t) = c(t - a) + O((t - a)^3).$$

From $rr' = x \cdot x' = c^2(t - a) + O((t - a)^3)$ follows

$$r'(t) = c + O((t - a)^2).$$

Hence $r(a + 0) = 0$ and $r'(a + 0) > 0$.
Similarly $r(b - 0) = 0$ and $r'(b - 0) < 0$. This allows Sturm's theorem to be applied on (a, b).

Since $e(t) = x(t)/r(t)$

$$e(t) = \frac{x'(a)}{c} + O((t - a)^2), \qquad a < t < b$$

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