PROPERTIES OF SUBHARMONIC FUNCTIONS IN THE HALF-PLANE

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1. **Introduction.** Recently Ahlfors and Heins [1] and Lelong-Ferrand have obtained the following theorem.

Theorem 1. *If $u(z)$ is a negative subharmonic function in $Rz > 0$, and* lim $\sup_{z\to 0} u(z)/Rz = \alpha > -\infty$ *in* $Rz > 0$, *then* $\lim_{r\to 0} u(re^{i\theta})/r\cos\theta = \alpha$ *for* $|\theta| < \pi/2$ *with the exception at most of a set of outer logarithmic capacity zero: uniformly in any closed interior angle if r is excluded from a set of finite logarithmic length; and without exception in any interior angle in which $u(z)$ is harmonic, (the Ahlfors-Heins-Ferrand theorem.*

(The original theorem in [4] deals more generally with a positive superharmonic function in n-dimension.)

Let us express the conclusion of Theorem 1 by saying that $u(re^{i\theta})/r\cos\theta$ tends effectively to α. By the statement that $u(re^{i\theta})/r\cos\theta$ is effectively bounded we shall mean the same thing with boundedness replacing approach to a limit.

It will be interesting to see how much the behavior of $u(z)$ on the imaginary axis can be weakened without destroying the conclusion of Theorem 1. Recently Boas has studied the same thing when z tends to the infinity for the functions of exponential type [2].

Theorem 2. *If $u(z)$ is subharmonic in $Rz \geq 0$ including infinity and excepting the origin and satisfies*

$$\text{(a)} \qquad \liminf_{r\to 0} r\int_{-\pi/2}^{+\pi/2} |u(re^{i\varphi})| \cos\varphi \, d\varphi = 0$$

$$\text{(b)} \qquad \limsup_{z\to 0} u(z)/Rz = \beta > -\infty \quad \left(|\arg z| < \delta < \frac{\pi}{2}\right)$$

and

$$\text{(c)} \qquad \int_{\to 0}^{1} t^{-2}u^{+}(\pm it)\, dt$$

exist, then for $|\theta| < \pi/2$

$$\lim_{r\to 0} \frac{u(re^{i\theta})}{r\cos\theta} = \beta$$

under the same exceptions as those of Theorem 1.

Theorem 3. *If $u(z)$ is subharmonic in $Rz \geq 0$ including infinity and ex-*

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