EXPANSIONS OF q-BERNOULLI NUMBERS

BY L. CARLITZ

1. The writer [3] has defined

$$\beta_n(x, q) = (q - 1)^{-n} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{r+1}{[r+1]} q^{rx}, \tag{1.1}$$

where

$$[r] = (q^r - 1)/(q - 1);$$

in particular

$$\beta_n(q) = \beta_n(0, q) = (q - 1)^{-n} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{r+1}{[r+1]}. \tag{1.2}$$

Also it was proved that

$$\lim_{q=1} \beta_n(q) = B_n , \tag{1.3}$$

where B_n is the ordinary Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Since $\beta_n(q)$ is a rational function of q, we may put

$$\beta_n(q) = \sum_{r=n}^{\infty} \beta_{nr}(q - 1)^{r-n}, \tag{1.4}$$

where the β_{nr} are rational numbers independent of q; in particular $\beta_{nn} = B_n$. The coefficients β_{nr} can be determined as follows.

If we put

$$\frac{y}{(1 + \lambda y)^{1/\lambda} - 1} = \sum_{k=0}^{\infty} b_k(\lambda) \frac{y^k}{k!} , \tag{1.5}$$

then it is not difficult to show that [1]

$$b_k(\lambda) = \sum_{s=0}^{k} B_s \lambda^{k-s} \sum_{j=s}^{k} \frac{(-1)^{k-j}}{j+1} \binom{j+1}{s} S(k, k - j), \tag{1.6}$$

where

$$x(x - 1) \cdots (x - k + 1) = \sum_{j=1}^{k} (-1)^{k-j} S(k, k - j) x^j. \tag{1.7}$$

Received October 12, 1957.