EXTENSION OF FUNCTIONS FROM DENSE SUBSPACES

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Introduction. We present here a systematic development of the theory of continuous extensions from dense subspaces of completely regular Hausdorff (crH) spaces. In some instances (notably §1 and Proposition 3.2), results are presented for spaces satisfying less restrictive separation axioms, in order to bring out certain topological features. Otherwise, no attempt has been made to indicate where separation hypotheses can be slightly relaxed; in general, we use essential properties of crH spaces, and removing this hypothesis requires separate special techniques.

In §1 we show that if a function can be continuously extended, the extension must have a certain explicit form. In §2, we apply the results of §1 to the problem of extending a function continuously over subspaces of βX (the Stone-Čech compactification of X). In case the range space is compact, this procedure yields quite simple proofs of well-known results of Stone [4] and Čech [2] (Corollary 2.2 below). The section concludes with solutions to some other problems in this area. It might be pointed out that §2, particularly 2.3, represents the essential content of [5] and [6]. (Though these papers deal with T_1-spaces, the conditions imposed on the functions are such that the proofs in §2 go through almost without modification.)

§3 deals with the problem of continuous extensions from a dense subspace of an arbitrary crH space. We show that this problem is closely connected with the problem covered in §2. Necessary and sufficient conditions are given for the existence of a continuous extension; we show that the possibility of extending over certain points of βX must be considered. As in §2, the problem becomes much simpler if the range space is compact. In this case it is rather easy to construct theorems on continuous extensions—one merely adjoins conditions guaranteeing the fulfillment of (ii), Corollary 3.5. Finally, in Theorem 3.8, we state precisely how far it is possible to extend a function continuously.

1. **Notation and preliminary theorems.** Throughout this paper, X and Y will be topological spaces, and φ will be a continuous mapping from X into Y. Let X be dense in a space E. Then with each $y \in Y$, we associate a set $E_y \subset E$, defined by

(a) $$E_y = \bigcap \mathrm{cl}_E\, \varphi^{-1}(V),$$

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