FUNDAMENTAL SOLUTIONS OF A LINEAR ORDINARY DIFFERENTIAL EQUATION OF THE THIRD ORDER IN THE NEIGHBORHOOD OF A SIMPLE SECOND ORDER TURNING POINT

By L. R. Bragg

1. **Introduction.** This paper is concerned with a determination of fundamental solutions, valid over a compact set $\tilde{C}$ of the complex z-plane, of a special class of differential equations of the form

$$y'''(z) + h_1(z, \lambda)y''(z) + \lambda^2 h_2(z, \lambda)y'(z) + \lambda^3 h_3(z, \lambda)y(z) = 0. \tag{1.1}$$

The complex parameter λ is assumed large in absolute value, and the h_i (z, λ) are series of the form

$$h_i(z, \lambda) = \sum_{j=0}^{\infty} h_{ij}(z)\lambda^{-j}, \tag{1.2}$$

the $h_{ij}(z)$ being analytic in z throughout $\tilde{C}$. The structure of $\tilde{C}$ will be described below. As is well-known (see [2]), the solutions of (1.1) depend upon the configuration in the γ-plane of the roots of its auxiliary equation

$$\gamma^3 + h_{10}(z)\gamma^2 + h_{20}(z)\gamma + h_{30}(z) = 0. \tag{1.3}$$

The problem considered here is that of describing the initial segments of a fundamental set of asymptotic solutions of type (1.1) over a compact set $\tilde{C}$ in which (a) two and only two roots of (1.3) meet at a single point such that (b) the discriminant of (1.3) vanishes to the second degree. The point at which this occurs is called a *simple turning point of the second order*. An analogous problem for a second order differential equation has been solved by R. W. McKelvey [6]. The solutions of (1.1) have been established [3] in the neighborhood of a point at which (a) holds but at which the discriminant of (1.3) vanishes to the first degree. Solutions are also known in the vicinity of a point where three roots of the auxiliary equation meet at a single point at which any two of them have contact of degree one [4]. For other configurations of the roots of the auxiliary equation, little is known [1].

The method that has proved fruitful in establishing results in the cases cited above is that of the related equation. By a related equation is meant a differential equation which approximates the given equation, say (1.1), and whose solutions can be given explicitly. In a recent paper [5], Langer has shown that

Received June 18, 1957; presented to the American Mathematical Society, September 2, 1955. This is the author's Ph.D. thesis accepted by the University of Wisconsin. This research was supported by the Wisconsin Alumni Research Foundation. The author wishes to thank Professor Langer for his guidance in the preparation of this paper.