DIFFERENCE EQUATIONS OF POLYHARMONIC TYPE

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1. **Introduction.** Of concern in this report are functions defined at the lattice points of a Euclidean space. These are the points whose coordinates are integers. The main equations to be considered arise from the application of the discrete harmonic operator D, and its powers. D corresponds to the Laplace differential operator Δ. Many of the properties of D have been discussed in detail by Courant, Friedrichs and Lewy [3], H. A. Heilbronn [7], R. J. Duffin [6], and others. The powers of D have been briefly mentioned in [3].

In two dimensions, continuous harmonic functions have a multiplicative closure which is best examined by the introduction of analytic functions. Isaacs [9] and Duffin [5], have discussed discrete analytic functions and a modified multiplication which gave closure. In any number of dimensions, the operators $r \cdot \text{grad}$ and the components of $r \times \text{grad}$ produce continuous harmonic functions when applied to a continuous harmonic function. Here we are interested in obtaining analogous operators for discrete harmonic functions.

McCrea and Whipple determined properties of the fundamental solution of the discrete harmonic equation [10]. In references [5] and [6], further properties were obtained by use of an operational calculus method based on Fourier series. The operational method is here employed for the generation of a fundamental solution of the discrete polyharmonic equation. Because we specify the asymptotic behavior of the fundamental solution, we call it the free space Green's function and refer to it as merely the Green's function.

Imitating continuous theory, we construct discrete polyharmonic functions from discrete polyharmonic functions of lower order. This enables us to evaluate the two-dimensional discrete biharmonic Green's function from the discrete harmonic Green's function which has been previously evaluated, [10]. Furthermore, we are able to give a numerical evaluation of the three-dimensional discrete harmonic Green's function. This is a stepwise evaluation in terms of three fundamental constants.

It is found that discrete polyharmonic functions defined on convex domains can be continued indefinitely. This result aids in giving a representation of discrete polyharmonic functions in terms of discrete harmonic functions.

It was shown by Heilbronn [7], that a two-dimensional discrete harmonic function defined on a finite convex domain can be represented by a harmonic polynomial. Here this theory is extended to the polyharmonic case, and a more general study is made of polynomial solutions. By Fourier methods,

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