# AVERAGES OF CONTINUOUS FUNCTIONS ON COMPACT SPACES

By Walter Rudin

**Introduction.** In this paper, $X$ will always denote a non-empty compact Hausdorff space, and $C(X)$ will be the Banach space of all real-valued continuous functions on $X$. The set of all integers will be denoted by $J$, and $N$ will stand for the set of all positive integers; $N$ will sometimes be considered a topological space, with the discrete topology.

The problem under consideration is the convergence of averages of the form

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i) \tag{1}$$

where $x_i \in X$ and $f \in C(X)$. Of particular interest is the case in which the sequence $\{x_i\}$ is generated by a homeomorphism $h$ of $X$, in the sense that $x_i = h(x_{i-1})$. Some positive results in this direction are summarized in Part I. The discussion of Theorems 1 and 2 does not differ much from that in [8; 118–9], although the emphasis there is entirely on metric spaces. I have not found Theorem 3 stated in the literature; but the essential part of the proof (the existence of a minimal compact invariant set) is known [3; Theorem 2.22].

In a paper on Banach limits of bounded sequences [4; 83], Jerison made a conjecture concerning the convergence of averages of the above sort if $X$ is $\beta N - N$ ($\beta N$ is the Čech compactification of $N$). The conjecture is stated and disproved in Part III. Theorem 6, which goes considerably beyond a mere disproof of Jerison's conjecture, is the principal result of the present paper. Part II contains a description of those properties of $\beta N$ which are used in the proof of Theorem 6.

Some questions concerning topological properties of compact Hausdorff spaces, which are suggested by the preceding results, are discussed in Part IV.

## 1. Some positive results.

1.1. Let $h$ be a homeomorphism of $X$ onto $X$. For $f \in C(X)$ and $x \in X$, define

$$s_n(f; x) = \frac{1}{n} \sum_{i=0}^{n-1} f(h^i(x)) \qquad (n \in N), \tag{2}$$

where $h^i(x) = h(h^{i-1}(x))$.

Pick $x_0 \in X$. For each $n$, the mapping $f \rightarrow s_n(f; x_0)$ is a bounded linear functional of norm 1 on $C(X)$. Since the unit sphere in the conjugate space of $C(X)$ is weakly compact, this sequence of functionals has a weak limit point (there may be many limit points, but we focus our attention on one), and by

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