TRANSFORMS OF TAUBERIAN SERIES BY RIESZ METHODS OF DIFFERENT ORDERS

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1. **Introduction.** The Riesz transform $R_n^{(r)}$ of order r, of a series $u_1 + u_2 + \ldots$ is defined by

$$(1.1) \qquad R_n^{(r)} = \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right)^r u_k , \qquad (n = 1, 2, 3, \cdots).$$

It is our purpose to study the relations between the Riesz transforms of real non-negative orders p and q, of series satisfying the Tauberian condition

$$(1.2) \qquad \limsup_{n\to\infty} n \mid u_n \mid < \infty .$$

We shall assume at all times that p and q are real numbers, not necessarily integers, for which $0 \leq p \leq q$. For each $\alpha > 0$, let $m(\alpha)$ and $n(\alpha)$ be positive integers, and let $m(\alpha) \to \infty$ and $n(\alpha) \to \infty$ as $\alpha \to \infty$. For convenience, we shall frequently write m, n for $m(\alpha)$, $n(\alpha)$. There is then a least constant B, dependent upon the constants p and q and the functions $m(\alpha)$ and $n(\alpha)$, such that

$$(1.3) \qquad \limsup_{\alpha\to\infty} \mid R_n^{(q)} - R_m^{(p)} \mid \leq B \limsup_{n\to\infty} n \mid u_n \mid,$$

whenever $\sum u_n$ is a series for which (1.2) holds. The constant B may be sometimes finite and sometimes $+\infty$. In the latter case we consider the right member of (1.3) to be $+\infty$ even when $\lim n \mid u_n \mid = 0$. We shall first determine the manner in which B depends upon p, q and the functions $m(\alpha)$ and $n(\alpha)$. We shall then suppose that p and q are fixed, and characterize the particular functions $m(\alpha)$ and $n(\alpha)$ for which B assumes its least possible value, designated by H_{pq}. We shall in addition solve other related problems.

An extensive bibliography of the field in which our problems lie has been given by Agnew [3]. Some of the problems in which we are interested were solved by Agnew [2], [3] for the case in which $p = 0$ and $q \geq 0$. Garten [5] solved similar problems, in which Riesz transforms are replaced by Cesàro transforms, for the case in which $p = 0$ and q is a positive integer. After the above cited results of Agnew and Garten were obtained, Agnew [4] found that if $\sum u_n$ satisfies (1.2), and if $r \geq 0$, then the Riesz transform $R_n^{(r)}$ and the Cesàro transform $C_n^{(r)}$ of order r are equiconvergent in the sense that

$$(1.4) \qquad \lim_{n\to\infty} (R_n^{(r)} - C_n^{(r)}) = 0.$$

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