SERIES OF LEGENDRE AND LAGUERRE POLYNOMIALS

By V. F. Cowling

1. **Introduction.** It is known [9; 238] that the domain of natural convergence of a series of Legendre polynomials $\sum a_n P_n(z)$ (unless otherwise indicated all summations run from 0 to ∞) is an ellipse with foci at -1 and $+1$. The sum, R, of the semi-major and semi-minor axes of this ellipse is given by $R = \liminf_n | a_n |^{-1/n}$. This formula for R is, of course, the analogue of the Cauchy-Hadamard formula for the radius of convergence of a power series. In order that the ellipse not degenerate into the segment of the real axis from -1 to $+1$ it is clearly sufficient that $R > 1$. In the case of series of Laguerre polynomials $\sum a_n L_n(z)$ the domain of natural convergence is a parabola (8) and [9; 246] having its focus at the origin and its latus rectum of length $2p^2$ where $p = -\limsup_m (2n^{\frac{1}{2}})^{-1} \log | a_n |$. Thus if $\epsilon > 0$ is given

$$(1.1) \qquad | a_n | \leq k(\epsilon) \exp \{2n^{\frac{1}{2}}(-p + \epsilon)\}$$

for $n > n_0(\epsilon)$.

In the case of power series and certain other classes of series there exists an extensive literature dealing with the relationship between the sequence of coefficients and singular points of the function defined by the series (6). One method for accomplishing this aim (2), (3) and (5) has been to regard the terms of the series $\sum a_n g_n(z)$ as the values assumed by an analytic function at the positive integers. Then in terms of the rate of growth, in certain directions, of an analytic function generating the coefficients, a region of regularity is determined which will under circumstances extend beyond the natural region of convergence. An attempt to apply such a procedure directly to series of Legendre or Laguerre polynomials, however, would involve the Legendre and Laguerre functions. Unfortunately the representations of these functions which are not too involved are valid only in limited portions of the plane, and their rates of growth as functions of $w(P_w(z)$ or $L_w(z))$ do not lend themselves readily to the sort of calculations required in (2), (3) or (5). This difficulty is to some extent circumvented by adapting a device due to Desaint [4] to the series under consideration. This results in Theorem 2.1. The results we obtain in the case of Legendre polynomials could easily be extended to series of general Jacobi polynomials. However, for the sake of simplicity we regard Legendre polynomials as sufficiently representative of the general problem. Analogous results are obtained for series of Laguerre polynomials of order zero (Theorem 3.1).

Received February 9, 1957; in revised form, November 4, 1957. This research was supported in part by a grant from the National Science Foundation.