AN EXTENSION OF E. HOPF'S MAXIMUM PRINCIPLE WITH AN APPLICATION TO RIEMANNIAN GEOMETRY

By E. Calabi

1. **Introduction.** There are numerous instances in function theory, partial differential equations, and differential geometry where one needs to apply a maximum principle. One complete statement of this principle, applied to twice differentiable functions, is due to E. Hopf [2] and quoted below as Theorem 1. In the course of recent investigations, whose results now are being prepared for publication, it has been necessary to prove Hopf's theorem in the case of functions that are not necessarily differentiable, but for which one verifies a certain weak, elliptic differential inequality. In the next section we give a precise definition of the inequality in question, showing that, if applied to twice differentiable functions, it reduces to the elementary notion of that inequality, and that in the extended sense it provides a more general condition under which E. Hopf's theorem is satisfied.

In §3 we review some known facts about geodesics in a Riemannian manifold, which are then applied in §4. In this last section we prove that in a Riemannian manifold whose Ricci curvature is non-negative, if r denotes the (shortest) geodesic distance from a fixed point $\mathbf{p}_0$, and Δ is the Laplace-Beltrami operator, the following weak, elliptic differential inequality is valid,

$$\Delta r \quad \text{(weakly)} \leq \frac{n-1}{r}, \tag{1.1}$$

and some geometrical and potential-theoretic conclusions are derived from this result. A simplified proof of our generalization of Hopf's theorem, actually reduced to a modification of the original proof by Hopf, was suggested by J. B. Serrin, whose help in this connection I wish to acknowledge gratefully.

2. **The maximum principle.** In this article we denote by L a linear, uniformly elliptic, second order differential operator in n dimensions, containing only first and second partial derivative terms: in other words, if $u = f(x)$ $((x) = (x^1, \cdots, x^n))$ is a function of n variables of class $\mathcal{C}^2$, then

$$L[u] = a^{ij}(x) \frac{\partial^2 u}{\partial x^i \, \partial x^j} + b^i(x) \frac{\partial u}{\partial x^i}, \tag{2.1}$$

where we make use of the summation convention. The only assumptions we make about the coefficients a^{ij} and b^i are that in a neighborhood of each point

Received May 1, 1957. This article was prepared while the author was partly supported by a research grant from the Office of Naval Research.