KERNEL FUNCTIONS AND EIGENFUNCTION EXPANSIONS

By Edward Nelson

1. **Introduction.** The problem of eigenfunction expansions is essentially that of exhibiting the spectral resolution of an operator in concrete analytical terms. In 1910 Hermann Weyl [20] achieved this for self-adjoint second-order ordinary differential operators. M. H. Stone [18] discussed this result in the light of the spectral theorem for self-adjoint operators on an abstract Hilbert space, and the result was extended to ordinary differential operators of higher order in varying degrees of generality by Coddington [6], Kodaira [13], in a forthcoming paper by Stone, and by others. Titchmarsh [19] gave explicit formulas for the spectral projections in terms of the Green's function, in the second-order case. In 1952 F. I. Mautner [14] introduced a method of defining eigenfunctions in terms of Radon-Nikodym derivatives, for a certain class of operators on L^2 spaces. This technique was used by Gårding [7], [8] and Browder [4] to establish eigenfunction expansions for elliptic partial differential operators, by Hörmander [12] for a class of partial differential operators with constant coefficients which are not elliptic in the usual sense, and by Bade and Schwartz [2]. A version of Radon-Nikodym derivatives for reflexive Banach spaces due to I. M. Gelfand [9] has been used by Gelfand and Kostyucenko [10] and by Browder [5] for general partial differential operators. In this case the "eigenfunctions" obtained are distributions.

Our purpose in this note is to establish eigenfunction expansions in a more constructive and explicit fashion, and in such a way that the results will apply to Hilbert spaces with norms other than the L^2 norm (for instance, to spaces of functions f with a norm involving various derivatives of f). We shall deal mainly with a self-adjoint operator S such that the resolvent $(S - l)^{-1}$, $\mathrm{Im}(l) \neq 0$, has a kernel function $G(x, y; l)$ which we assume known. The kernel of the spectral projections may then be determined by a sort of contour integral of G (Theorem 9). This is similar to a formula of Titchmarsh [19] in the second-order ordinary differential case. The kernels $E(x, y; \Delta)$ of the spectral projection $E(\Delta)$, for Δ a bounded real Borel set, are positive definite functions. Regarded as a function of Δ, this gives a measure whose values are positive definite functions. For a separable Hilbert space there is a numerical measure ρ such that $E(\Delta) = 0$ when $\rho(\Delta) = 0$. Extending the Radon-Nikodym derivative to this situation (Theorem 1), we obtain a family of positive definite functions $e(x, y; \lambda)$, $-\infty < \lambda < \infty$, such that $E(x, y; \Delta) = \int_\Delta e(x, y; \lambda)d\rho(\lambda)$ for all bounded Borel sets Δ. Now every positive definite function is the reproducing kernel of a Hilbert space of functions. In particular $e(x, y; \lambda)$ is the reproducing kernel of a space H_λ. It is then easily shown that the original

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