CONGRUENCE PROPERTIES OF THE CLASSICAL ORTHOGONAL POLYNOMIALS

BY W. A. AL-SALAM AND L. CARLITZ

1. **Introduction.** In an earlier paper [1] the writers derived finite summation formulas for the classical orthogonal functions analogous to certain integral formulas. For example

$$\frac{1}{t}\sum_{r=0}^{t-1}\left\{x+(x^2-1)^{\frac{1}{2}}\cos\left(\phi+\frac{2r\pi}{t}\right)\right\}^n = P_n(x)+2\sum_{1\leq st\leq n}\frac{n!}{(n+st)!}P_n^{st}(x)\cos s\phi, \tag{1.1}$$

where n and t are arbitrary positive integers, $P_n(x)$ is the Legendre polynomial, and $P_n^s(x)$ is the associated Legendre function.

Another example is

$$\begin{aligned}2^{(m+n)/2}\frac{1}{t}\sum_{r=0}^{t-1}H_{m+n}&\left[x\left\{1+\sec\left(\phi+\frac{2r\pi}{t}\right)\right\}^{\frac{1}{2}}\right]\\ &\cdot\cos^{\frac{1}{2}(m+n)}\left(\phi+\frac{2r\pi}{t}\right)\cos^{\frac{1}{2}(m-n)}\left(\phi+\frac{2r\pi}{t}\right)\\ &=\sum_k\frac{(m+n)!}{(m+kt)!(n-kt)!}H_{m+kt}(x)H_{n-kt}(x)\cos kt\phi,\end{aligned} \tag{1.2}$$

where m and n are arbitrary non-negative integers, and $H_n(x)$ denotes the Hermite polynomial.

In the present paper we find congruential analogs of most of the results of [1]. Let $p = 2w + 1$ be an odd prime. Then, for example, corresponding to (1.1) and (1.2) we have

$$\begin{aligned}-\sum_{r=1}^{p-1}\{x+\tfrac{1}{2}(x^2-1)^{\frac{1}{2}}(r+r^{-1})\}^n &\equiv P_n(x)\\ &+2\sum_{1\leq 2wk\leq n}\frac{n!}{(n+2wk)!}P_n^{2wk}(x)\pmod{p},\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}-\sum_{r=1}^{p-1}H_{m+n}&\left\{x\left(1+\frac{2}{r^2+r^{-2}}\right)\right\}(r^2+r^{-2})^{\frac{1}{2}(m+n)}(r^{m-n}+r^{n-m})\\ &\equiv 2\sum_k\binom{m+n}{m+kw}H_{m+kw}(x)H_{n-kw}(x)\pmod{p}.\end{aligned} \tag{1.4}$$

Similar formulas are found for the Jacobi and Laguerre polynomials.

Received June 13, 1957.