MATRIX NORMS AND VECTOR MEASURES

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A number of writers have laid down definitions for some sort of measure (variously called a *norm*, the term we shall use in this paper, a *modulus* or a *bound*) for a matrix (not necessarily square) or a linear transformation (not necessarily finite). These norms may be related to possible measures for the vectors on which they operate. It is the purpose of this paper to show the relations between the various systems that have been proposed, to reduce the numbers of postulates by showing that some may be derived from the rest, and to demonstrate certain relations between measures and norms.

1. Bowker (1947), Dwyer and Waugh (1953) and Wong (1954) have defined, under various names, quantities we shall call matrix *norms*, with the following properties common to all definitions:

If A is a matrix (not necessarily square or of finite order)

(I). $N(A) \geq 0$; *and* (Ia) $N(A) = 0$ if and only if $A = 0$;

(II). $N(sA) = | s | \cdot N(A)$, where s is a scalar quantity;

(III). $N(AB) \leq N(A) \cdot N(B)$;

(IV). $N(A + B) \leq N(A) + N(B)$.

We shall show that Ia may be derived from I, II, and IV, except for the trivial case in which $N(A) = 0$ for every A.

Wong further includes as a postulate

(V). If $\lim_{p \to \infty} A_p = A$, then $\lim_{p \to \infty} N(A - A_p) = 0$.

We shall show that this may be derived from II and IV if A is of finite order. It does not follow generally if A is of infinite order.

Bowker and Dwyer and Waugh have a further postulate

(VIa). If E_{ij} is the matrix for which the (i, j) element is 1 and the rest zero, then $N(E_{ij}) = 1$ for all i, j.

Alternatively, Wong has

(VIb). $N(I) = 1$ for I the unit matrix of any order,

and

(VIc). If E is a submatrix of some I, $N(E) \leq 1$.

Wong also has

(VII). If $A \geq 0$, $B \geq 0$, $N(A + B) \geq \max \{N(A), N(B)\}$.

(Throughout this paper, $A \geq 0$ will be used to denote $a_{ij} \geq 0$ for all i, j.) Since, however, if E in VIc is a submatrix of I_n, so is $I_n - E$, VIc follows immediately from VIb and VII. In fact, we shall show that $N(E) < 1$ cannot occur.

Lonseth (1947) starts instead from a vector space in which there is a measure

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