COMPACT RINGS WITH OPEN RADICAL

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1. **Introduction.** Zelinsky has a neat characterization of compact totally disconnected rings as the inverse limits of finite rings [9]. We think it might be of interest to investigate such compact rings in the case that all the finite rings in the inverse system have the same semi-simple part. That is the case when the compact ring has an open radical.

In §2, we consider a Wedderburn-Malcev Theorem for compact rings with open radical. In §3, we investigate a condition under which the powers of the radical give a base for the topology in the ring. We then obtain a prototype for such a ring, a "free" compact ring. We show in the following section that the free compact rings of §4 have certain nice segregation properties.

Throughout the paper we will work with compact totally disconnected rings. In §§4 and 5, we make the stronger assumption that the ring is a compact ring with identity.

2. **Wedderburn decompositions.** In this section we consider a Wedderburn decomposition of a compact ring R with open (Jacobson) radical N.

DEFINITION. R admits a *Wedderburn decomposition* if $R = S + N$ direct sum, where S is a closed semisimple subring. The subring S is a *Wedderburn factor* of R.

In the case of a finite ring R the characteristic of R and R/N might be different, in which case it is possible that there is no Wedderburn factor. This, however, is the only barrier to the existence of a Wedderburn factor. We show that a similar situation holds in a compact ring with open radical. If R is compact and N open, then the semisimple ring R/N is compact discrete, hence finite. Thus R/N has an identity $\bar{e}$ [1, Corollary 4.3B].

THEOREM 1. *If R is compact, N open and $\bar{e}$ the identity of R/N, then R admits a Wedderburn decomposition if and only if $\bar{e}$ can be raised to an element e of R having the same additive order as $\bar{e}$. If S and S' are two Wedderburn factors of R, then there exists $z \varepsilon N$ such that if z' is its quasi-inverse, then the mapping $s \rightarrow s - zs - sz' + zsz'$ is an isomorphism of S onto S'.*

Proof. The "only if" part is clear. Going the other way, let n be the additive order of $\bar{e}$ and R' the closed ideal $\{r \varepsilon R \mid nr = 0\}$. R' is a compact subring of R. The radical of R' is $R' \cap N$ which is open in R'. The image of R' in R/N is $R'/R' \cap N$. However, the latter is an ideal of R/N and, by assumption, contains $\bar{e}$. Thus $R'/R' \cap N = R/N$, and it is sufficient to find a Wedderburn factor for

Received March 23, 1957; in revised form, June 18, 1957.