OSCILLATORY SOLUTIONS OF NONLINEAR AUTONOMOUS DIFFERENTIAL EQUATIONS OF ORDER HIGHER THAN TWO

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Introduction. Consider the nonlinear autonomous differential equation of the second order

$$\text{(a)} \qquad \ddot{x} + f(x)\dot{x} + h(x) = 0.$$

It can be written as a system,

$$\text{(b)} \qquad \dot{x} = y, \qquad \dot{y} = -f(x)y - h(x).$$

Assume that the origin is the only critical point of the system, and that it is negatively asymptotically stable, that is,

$$xh(x) > 0(x \neq 0), \qquad f(0) < 0.$$

If $x(t)$ $(t \geq 0)$ is a bounded solution of (a) such that $\dot{x}(t)$ is also bounded, then $(x(t), y(t))$, where $y(t) = \dot{x}(t)$, is a bounded solution of (b), and from the general theory of Bendixon-Poincaré it follows that this solution must spiral asymptotically toward a periodic solution of (b). Since every periodic solution contains critical points, it follows that the solution $x(t)$ is oscillatory, that is, it has an infinite number of zeros.

Equation (a) is a special case of the n-th order differential equation

$$\text{(c)} \qquad \frac{d^n x}{dt^n} + \frac{d^{n-1}G_1(x)}{dt^{n-1}} + \frac{d^{n-2}G_2(x)}{dt^{n-2}} + \cdots + \frac{dG_{n-1}(x)}{dt} + G_n(x) = 0.$$

In this paper we shall prove that under certain assumptions on the $G_i(x)$, every solution of (c) which is bounded together with its first $n - 1$ derivatives, is oscillatory. We shall give the proof only for the case $n = 3$, and then state the general theorem.

1. The equation to be first considered is

$$(1) \qquad \dddot{x} + g(x)\ddot{x} + g'(x)\dot{x}^2 + f(x)\dot{x} + h(x) = 0.$$

A solution $x(t)$ $(0 \leq t < \infty)$ of (1) is said to be *strongly bounded* if $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ are bounded. Clearly, if the assumption 1) of Theorem 1 below is satisfied, then equation (1) is equivalent to the system

$$(2) \qquad \begin{aligned} \dot{x} &= y - G(x) \\ \dot{y} &= z - F(x) \\ \dot{z} &= -h(x), \end{aligned}$$

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