TENSOR PRODUCTS OF SEMIPRIMARY ALGEBRAS

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Several recent papers [2], [3] consider algebras A such that $A \otimes B$ is semiprimary (especially in the sense SP_3 below) for some B. In the present paper we point out that such conditions can have strong implications for the structure of A. We should also call attention to a by-product on rings with minimum condition, the corollary to Theorem 3.

Various definitions of the term "semiprimary" are in current use. Three of them that shall interest us give rise to the following three classes. In each case A is assumed to be an algebra over a fixed ring F. "Semisimple" is to mean semisimple with minimum condition.

$SP_1 = \{A \mid \text{for some quasiregular ideal } N, A/N \text{ is semisimple}\}$
$SP_2 = \{A \mid \text{for some nil ideal } N, A/N \text{ is semisimple}\}$
$SP_3 = \{A \mid \text{for some nilpotent ideal } N, A/N \text{ is semisimple}\}$.

Note that in each case N is the (Jacobson) radical of A. Perhaps we should add a fourth class which inspired all the others:
$SP_4 = \{A \mid A \text{ satisfies the minimum condition on left ideals}\}$.
Clearly $SP_i \subset SP_j$ when $i > j$.

Lemma 1. *If $A \in SP_i$ and α is a homomorphism defined on A, then $\alpha A \in SP_i$ (cf. [2, Proposition 1].).*

Proof. If N is a quasiregular, nil or nilpotent ideal in A, then αN has the same properties in αA. Moreover, $\alpha A/\alpha N$ is a homomorphic image of A/N and so is semisimple if A/N is.

Lemma 2. *Let A and B be two algebras over a field F such that $A \otimes B \in SP_1$. Then either B is quasiregular or $A \in SP_1$.*

Proof. If A is quasiregular, then $A \in SP_1$. Otherwise the radical N of A is the intersection of the regular maximal left A-ideals. Consider the class of all $P \otimes B$ where P is the intersection of a finite number of regular maximal left ideals in A. If R is the radical of $A \otimes B$, there is a $P \otimes B$ which is minimal modulo R. That is, for every regular maximal left ideal J in A,

$$P \otimes B \subset R + (P \cap J) \otimes B. \tag{1}$$

If $P \neq N$, choose J such that $P \cap J \neq P$; hence $P + J = A$. Now the regularity of J means that the irreducible A-module A/J contains a nonzero element m whose annihilator is J. If $P + J = A$ we can find $e \in P$ such that

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