SOME POLYNOMIALS RELATED TO THETA FUNCTIONS

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1. Let

$$H_n(x) = H_n(x, q) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} x^r, \tag{1}$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(q)_n}{(q)_r(q)_{n-r}}, \qquad (q)_n = (1-q)(1-q^2)\cdots(1-q^n), \qquad (q)_0 = 1.$$

Alternatively $H_n(x)$ can be defined by means of

$$\prod_{r=0}^{\infty} (1 - q^r t)^{-1}(1 - q^r x t)^{-1} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{(q)_n} \qquad (|q| < 1). \tag{2}$$

Polynomials closely related to $H_n(x)$ have been studied by Wigert [7] and Szegö [5]; see also Hahn [4]. The polynomial $H_n(x)$ is in some respects an analog of the Hermite polynomial. The writer [3] showed that

$$H_{m+n}(x) = \sum_{r=0}^{\min(m,n)} (-1)^r q^{\frac{1}{2}r(r-1)} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} (q)_r x^r H_{m-r}(x)\, H_{n-r}(x). \tag{3}$$

This result was proved by induction. It may be of interest to point out that (3) can be proved by a method analogous to that used by Burchnall [1] in proving the corresponding formula for Hermite polynomials.

Indeed if we define the operator E^n by means of

$$E^n f(x) = f(q^n x) \tag{4}$$

and Δ^n by means of

$$\Delta^n = (1 - E)(q - E) \cdots (q^{n-1} - E), \tag{5}$$

then it is easily verified that

$$E^n = \sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r \tag{6}$$

and

$$\Delta^n = \sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}(n-r)(n-r-1)} E^r. \tag{7}$$

Received February 28, 1957.