BOUNDS ON THE FOURIER TRANSFORMS OF MONOTONIC FUNCTIONS

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Let Φ be the class of real, integrable, monotonic decreasing functions $\varphi(t)$ on $(0, \infty)$ satisfying the conditions $\int_0^\infty \varphi(t)\,dt = 1$ and $\varphi(0) = 1$ and consider the Fourier cosine and sine transforms given by (1) and (2).

$$f(x) = \int_0^\infty \varphi(t) \cos xt\, dt \tag{1}$$

$$g(x) = \int_0^\infty \varphi(t) \sin xt\, dt. \tag{2}$$

It has been shown previously [5; Theorem 1] that if $\varphi(t)$ belongs to the class Φ, then the integral of $f(x)$ given by (3) is bounded by (4) and (5), and these expressions are best possible for all permissible functions in class Φ.

$$F(x) = \int_0^x f(\xi)\, d\xi = \int_0^\infty \frac{\varphi(t)}{t} \sin xt\, dt \tag{3}$$

$$0 \leq F(x) \leq Si(x) = \int_0^x \frac{\sin \xi}{\xi}\, d\xi \quad \text{for} \quad 0 \leq x \leq \pi \tag{4}$$

$$0 < F(x) < Si(\pi) \quad \text{for} \quad \pi < x. \tag{5}$$

The purpose of this paper is to improve the previously presented results on $f(x)$ [5; Theorem 2] so that they too are the best possible and to develop the lowest possible upper bounds on $g(x)$. The greatest lower bound on $g(x)$ is known [2; Theorem 123]. The proof of the existence of the extremal functions given by the lemma was suggested by Professor Harold S. Shapiro.

A similar problem has been considered by J. F. Steffensen [1] where, given a particular function $\varphi(t)$ in class Φ, bounds on $f(x)$ and $g(x)$ are obtained in terms of the values that $\varphi(t)$ assumes at discrete points of t. The results of this paper are distinct in that the best possible bounds on $f(x)$ and $g(x)$ are obtained for the entire class Φ and these bounds do not depend upon the values of any particular $\varphi(t)$ in Φ.

The following lemma will be needed.

Lemma. *Let the functional $Q(\varphi)$ be defined by*

$$Q(\varphi) = \int_0^\infty \varphi(t) R(t)\, dt$$

where $R(t)$ is a bounded, integrable function. Then $Q(\varphi)$ attains its maximum and its minimum as $\varphi(t)$ ranges through all the admissible functions in class Φ.

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