FUNCTIONS ON CIRCULAR SUBSETS OF THE SPACE OF n COMPLEX VARIABLES

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1. **Introduction.** A subset X of the space C^n of n complex variables will be called *circular* if whenever $(x_1, \cdots, x_n)$ in X and $(y_1, \cdots, y_n)$ in C^n satisfy $|y_i| = |x_i|$ for all i, the point $(y_1, \cdots, y_n)$ must be in X. This paper is devoted to the study of the Banach algebra $A(X)$ of complex valued functions on X that can be approximated uniformly on X by polynomials.

In §2 we show that for an arbitrary bounded X, every function in $A(X)$ has a natural continuation to the set

$$(1.1) \qquad C_P(X) = \{y: |f(y)| \leq \sup_{x \in X} |f(x)| \quad \textit{for all polynomials} \quad f\}$$

which can be identified in a natural manner with the maximal ideal space of $A(X)$. For a circular X, the main result of [7] shows that $C_P(X)$ is identical with

$$(1.2) \qquad C_M(X) = \{y: |m(y)| \leq \sup_{x \in X} |m(x)| \quad \textit{for all monomials} \quad m\}.$$

Section 3 is devoted to two characterizations of the functions occurring in $A(X)$ for X compact and circular. The first characterization is in terms of Fourier coefficients, the result reducing in the case of one complex variable and $X = \{w : |w| = 1\}$ to the well known fact that $A(X)$ consists of all continuous functions whose Fourier series is of the form $\sum_{n=0}^{\infty} a_n e^{in\theta}$. The second characterization is valid only for those X that contain with each point $(x_1, \cdots, x_n)$, all points $(y_1, \cdots, y_n)$ with $|y_i| \leq |x_i|$ for all i. The characterization is in terms of analyticity and reduces in the case of one complex variable and $X = \{w : |w| \leq 1\}$ to the also well known fact that $A(X)$ consists of all functions that are continuous on X and analytic at each interior point.

In §4 the Silov boundary (see [9; 80]) of $A(X)$ is studied. Under our identification of the maximal ideal space of $A(X)$ with the set $C_P(X)$, the Silov boundary becomes identified with the smallest closed subset of $C_P(X)$ on which each function in $A(C_P(X))$ attains its maximum modulus. An application of the main theorem of [7] shows that if X is circular, this set is also the smallest closed subset of $C_P(X)$ on which each monomial attains its maximum modulus. This allows us to use convexity arguments to show that the set can be characterized as the closure of the collection of those points in $C_P(X)$ that it seems most appropriate to call the *multiplicative extreme points*.

Section 5 is devoted to the establishment of a Cauchy type integral formula for the functions in $A(C_P(X))$ that reduces to the classical formula (see [2; 40]) if $X = \{(x_1, \cdots, x_n) : |x_r| \leq 1\}$. This formula expresses the value of any f

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