COMMENTS ON "FLAT" OSCILLATIONS OF LOW FREQUENCY

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Let D_f denote the differential equation $x'' + f(t)x = 0$ in which the coefficient function f is given, as real-valued and continuous, for large positive t. With reference to any fixed index α, let $f(t)$ or the corresponding D_f be called of class α (in symbols: $f \in \alpha$) if $x(t) = O(t^\alpha)$, where $t \to \infty$, holds for every solution $x(t)$ of D_f. By $f(\varepsilon)\alpha$ will be meant that $f \in \beta$ does or does not hold according as $\beta = \alpha$ or $\beta < \alpha$. Finally, $f \in^* \alpha$ will mean that every solution $x(t)$ of D_f, besides being $O(t^\alpha)$, has a derivative satisfying $x'(t) = O(t^{\gamma-1})$ for every $\gamma > \alpha$.

The index $\alpha = \frac{1}{2}$ is of particular interest. For it was pointed out in [4] that *D_f must be oscillatory if $f \in \frac{1}{2}$*, whereas D_f can be non-oscillatory if $f \in \alpha$ holds for every $\alpha > \frac{1}{2}$ only. The second of these assertions follows by observing that if $D(\mu)$ denotes the case $f(t) = \mu^2/t^2$ of D_f, where μ is a constant (cf. [2]), then the general solution of $D(\frac{1}{2})$ is a superposition of $x(t) = t^{\frac{1}{2}}$ and $x(t) = t^{\frac{1}{2}} \log t$.

If $0 < \mu < \frac{1}{2}$, then $D(\mu)$ is non-oscillatory and of class α, where $\alpha = \alpha(\mu)$ is the larger of the two roots of the quadratic equation $\alpha(\alpha - 1) + \mu^2 = 0$ (in fact, $x(t) = t^\alpha$ is a solution for either root). But if $\frac{1}{2} < \mu < \infty$, then the two roots are of the form $\alpha = \frac{1}{2} \pm i\lambda$, where $\lambda = \lambda(\mu)$ is positive (and, incidentally, $\lambda(\frac{1}{2} + 0) = 0$ and $\lambda(\infty) = \infty$); so that $D(\mu)$ becomes oscillatory, the general solution being a superposition of $t^{\frac{1}{2}} \cos(\lambda \log t)$ and $t^{\frac{1}{2}} \sin(\lambda \log t)$. Accordingly, if $f_\mu(t) = \mu^2/t^2$, where $\frac{1}{2} < \mu < \infty$, then $f_\mu \in \frac{1}{2}$ and, what is more, both $f_\mu(\varepsilon)\frac{1}{2}$ and $f_\mu \in^* \frac{1}{2}$ hold. It might appear unexpected that the *sharp* relation $f_\mu(\varepsilon)\frac{1}{2}$ is independent of the *numerical* value of μ $(> \frac{1}{2})$.

Consider now an unspecified $D_f : x'' + f(t)x = 0$ (so that $f(t) = \mu^2/t^2$ is not assumed) and an unspecified $D_g : y'' + g(t)y = 0$. Suppose that $f \in \frac{1}{2}$ and that $g(t)$ is so "close" to $f(t)$ for large t as to satisfy the following condition (C):

$$(C) \qquad \int^\infty t \mid f(t) - g(t) \mid dt < \infty .$$

Then, as $t \to \infty$, the solutions $x(t)$ of D_f and their derivatives $x'(t)$ are (for $t \to \infty$) in asymptotic one-to-one correspondence with the solutions $y(t)$ of D_g and their derivatives $y'(t)$. In fact, if $x_1(t)$, $x_2(t)$ is any pair of solutions $x(t)$ of D_f, then $f \in \frac{1}{2}$ and (C) imply that

$$\int^\infty (\mid x_1(t) \mid^2 + \mid x_2(t) \mid^2) \mid f(t) - g(t) \mid dt < \infty .$$

Hence, the assertion, concerning the asymptotic one-to-one correspondence, can be concluded from a general theorem on "small perturbations" [5].

Clearly, the result contains the following corollaries: $f \in \frac{1}{2}$ and (C) imply that $g \in \frac{1}{2}$, and $f(\varepsilon)\frac{1}{2}$ and (C) imply that $g(\varepsilon)\frac{1}{2}$, finally $f \in^* \frac{1}{2}$ and (C) imply that $g \in^* \frac{1}{2}$

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