CERTAIN SUBMODULES OF SIMPLE RINGS WITH INVOLUTION

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In this paper we continue the study of substructures of simple rings with involution which are invariant with respect to Lie products with the skew elements (or commutators thereof) or with respect to Jordan products with the symmetric elements.

A will, herein, always denote a simple ring possessing an involution*, K will denote the skew elements of A; that is, $K = \{x \in A \mid x^* = -x\}$ and S will represent the symmetric elements of A, that is, $S = \{x \in A \mid x^* = x\}$. It is assumed throughout the paper that the characteristic of A is different from 2 and that either Z, the center of A, is (0) or that A is more than 16-dimensional over Z.

If L, M are additive subgroups of A, then $[L, M]$ will be the additive subgroup of A generated by all the elements $lm - ml$ where $l \in L$, $m \in M$, and $L \circ M$ will be the additive subgroup of A generated by all the elements $lm + ml$ where $l \in L$ and $m \in M$.

1. Submodules of K invariant under Jordan products with S.

In this section we consider additive subgroups U, of K, such that $U \circ S \subset U$. For these we prove

Theorem 1.1. *Suppose that A is a simple ring, of characteristic not 2, and with center $Z = (0)$ or with the dimension of A over Z larger than 16; suppose further that $U \subset K$ satisfies $U \circ S \subset U$. Then either $U = (0)$ or $U = K$.*

Proof. Let $u \in U$, $s, t \in S$. By our hypothesis, $us + su \in U$. But then $(us + su)t + t(us + su) \in U$. Writing this out we have

$$ust + sut + tus + tsu \in U. \tag{1}$$

Interchanging s and t we obtain

$$uts + tus + sut + stu \in U. \tag{2}$$

Thus on subtracting (2) from (1) we arrive at

$$u(st - ts) - (st - ts)u \in U$$

for all $s, t \in S$ and all $u \in U$. However, by the results of [1] (see Theorem 9a) $[K, K] = [S, S]$, so the fact that $[U, [S, S]] \subset U$, which was established above, becomes

$$[U, [K, K]] \subset U. \tag{3}$$

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