## THE EXPANSIONS OF SOME INFINITE PRODUCTS

## By Richard Bellman

1. Introduction. The technique used by Euler, Gauss, and Jacobi to obtain identities of the form

$$
\begin{equation*}
\prod_{k=1}^{\infty}\left(1-x^{k}\right)=\sum_{n=0}^{\infty} \frac{(-1)^{n} x^{n(n+1) / 2}}{\prod_{k=1}^{n}\left(1-x^{k}\right)}, \quad|x|<1 \tag{1}
\end{equation*}
$$

was the following. We begin with the function

$$
\begin{equation*}
f(x, t)=\prod_{k=1}^{\infty}\left(1-x^{k} t\right), \quad|x|<1 \tag{2}
\end{equation*}
$$

and observe that it satisfies the functional equation

$$
\begin{equation*}
(1-x t) f(x, t x)=f(x, t) \tag{3}
\end{equation*}
$$

Writing

$$
\begin{equation*}
f(x, t)=\sum_{n=0}^{\infty} a_{n}(x) t^{n} \tag{4}
\end{equation*}
$$

the relation in (3) yields

$$
\begin{equation*}
\sum_{n=0}^{\infty} a_{n}(x) t^{n}=(1-x t) \sum_{n=0}^{\infty} a_{n}(x) x^{n} t^{n} \tag{5}
\end{equation*}
$$

whence, equating coefficients,

$$
\begin{equation*}
a_{n}(x)=a_{n}(x) x^{n}-a_{n-1}(x) x^{n} \tag{6}
\end{equation*}
$$

This leads to the formula

$$
\begin{equation*}
a_{n}(x)=\frac{(-1)^{n} x^{n(n+1) / 2}}{\prod_{k=1}^{n}\left(1-x^{k}\right)} \tag{7}
\end{equation*}
$$

If we attempt to follow the same method for the product

$$
\begin{equation*}
f(x, y)=\prod_{k, l=0}^{\infty}\left(1-x^{k} y^{l}\right) \tag{8}
\end{equation*}
$$

where the prime indicates that $k$ and $l$ are not simultaneously zero, we encounter a difficulty. Setting

$$
\begin{equation*}
f(x, y, t)=\prod_{k, l=0}^{\infty}\left(1-x^{k} y^{l} t\right) \tag{9}
\end{equation*}
$$

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