THE EXPANSIONS OF SOME INFINITE PRODUCTS

By Richard Bellman

1. **Introduction.** The technique used by Euler, Gauss, and Jacobi to obtain identities of the form

$$\prod_{k=1}^{\infty} (1 - x^k) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n(n+1)/2}}{\prod_{k=1}^{n} (1 - x^k)}, \qquad |x| < 1, \tag{1}$$

was the following. We begin with the function

$$f(x, t) = \prod_{k=1}^{\infty} (1 - x^k t), \qquad |x| < 1, \tag{2}$$

and observe that it satisfies the functional equation

$$(1 - xt) f(x, tx) = f(x, t). \tag{3}$$

Writing

$$f(x, t) = \sum_{n=0}^{\infty} a_n(x) t^n, \tag{4}$$

the relation in (3) yields

$$\sum_{n=0}^{\infty} a_n(x) t^n = (1 - xt) \sum_{n=0}^{\infty} a_n(x) x^n t^n, \tag{5}$$

whence, equating coefficients,

$$a_n(x) = a_n(x) x^n - a_{n-1}(x) x^n. \tag{6}$$

This leads to the formula

$$a_n(x) = \frac{(-1)^n x^{n(n+1)/2}}{\prod_{k=1}^{n} (1 - x^k)}. \tag{7}$$

If we attempt to follow the same method for the product

$$f(x, y) = \prod_{k,l=0}^{\infty} (1 - x^k y^l), \tag{8}$$

where the prime indicates that k and l are not simultaneously zero, we encounter a difficulty. Setting

$$f(x, y, t) = \prod_{k,l=0}^{\infty}{}' (1 - x^k y^l t), \tag{9}$$

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