CONVEX FUNCTIONS OF QUADRATIC FORMS

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The purpose of this paper is to obtain some extreme value results for convex functions of Hermitian forms (Theorem 1). In Theorem 2 we investigate conditions under which extremal sets span invariant subspaces. In the remainder of the paper we generalize some recent results of K. Fan [4, 5] and F. Rellich [9].

Let $\lambda^{(1)}, \cdots, \lambda^{(k)}$ be fixed real n-vectors,

$$\lambda^{(j)} = (\lambda_1^{(j)}, \cdots, \lambda_n^{(j)}) \qquad j = 1, \cdots, k \leq n$$

and let

$$I_j = [\min_t \lambda_t^{(j)}, \quad \max_t \lambda_t^{(j)}].$$

Let R be defined as the Cartesian product

$$R = I_1 \times \cdots \times I_k .$$

By Ω_n we shall denote the convex hull of all n-square permutation matrices. This is known to be the polyhedron of all n-square doubly stochastic (d.s.) matrices [1, 2]. If S is any n-square complex matrix, then $S(i)$ will denote the i-th row of S. Let $f(t) = f(t_1, \cdots, t_k)$ be defined and bounded on R and let σ be a $1-1$ function on the integers $1, \cdots, k$ to the integers $1, \cdots, n$; we define a pair of numbers associated with f as follows:

$$m = \min_\sigma f(\lambda_{\sigma(1)}^{(1)}, \cdots, \lambda_{\sigma(k)}^{(k)})$$

$$M = \max_\sigma f(\lambda_{\sigma(1)}^{(1)}, \cdots, \lambda_{\sigma(k)}^{(k)}).$$

Now let A_j, $j = 1, \cdots, k$, be complex n-square Hermitian matrices which commute pairwise. It is known that the A_j have a common set of orthonormal (o.n.) eigenvectors $u_1, \cdots, u_n$ and we choose our notation so that

$$A_j u_i = \lambda_i^{(j)} u_i \qquad i = 1, \cdots, n$$

$$j = 1, \cdots, k.$$

That is, $\lambda_i^{(j)}$ $i = 1, \cdots, n$ are the eigenvalues of A_j.

Theorem 1. *If f is convex on R then*

$$\max_{(x_i, x_j) = \delta_{ij}} f((A_1 x_1, x_1), \cdots, (A_k x_k, x_k)) = M$$

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