RELATIVELY COMPLETE FIELDS

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In this note we discuss some properties of relatively complete fields. According to Ostrowski, a field is termed relatively complete if Hensel's lemma holds with respect to the given non-Archimedean valuation. (Throughout this note "valuation" means "non-Archimedean valuation.") There are many fields which are not complete but relatively complete. To find examples of such fields it suffices to consider the absolutely algebraic subfield of the field of p-adic rationals, or infinite algebraic extensions of a field which is complete with respect to rank one valuation. (See for example [6].) We establish first that relatively completeness is equivalent to the uniqueness of the extension of the given valuation which gives in turn an alternative proof for Hensel's lemma, avoiding approximation process. (It is well known that relative completeness implies the unique extension of the valuation to the algebraic closure but the reverse statement seems to have appeared in no publication.) We generalize the result obtained by Kaplansky-Schilling in [4]; namely, we prove that if K is relatively complete with respect to a rank one valuation and is not algebraically closed, every subfield k with K/k finite algebraic is relatively complete with respect to the induced valuation. As for the case when K is algebraically closed, we give a criterion for the question.

Theorem 1. *A field is relatively complete with respect to a valuation V if and only if V has a unique extension to its algebraic closure.*

Proof. Whenever k is relatively complete with respect to a valuation V, it is well known that V is uniquely extended to its algebraic closure, namely, its extension is given by $V(A) = | N_{K/k}A |^{1\deg/K/k}$, where $A \varepsilon K$, for all $A \varepsilon k^{\mathrm{clos}}$. Now conversely, suppose V has unique extension to its algebraic closure. Then for all $A \varepsilon k^{\mathrm{clos}}$, all $\sigma \varepsilon G(k^{\mathrm{clos}}/k)$, we have $V(A^{\sigma}) = V(A)$. Now if a monic polynomial $f(x)$ in $o[x]$ (where o is the valuation ring with respect to the given valuation V) factors into a product of relatively prime factors in its residue class field, namely, $[f(x)] = [g(x)]\,[h(x)]$ with $([g(x)], [h(x)]) = 1$ in the residue class field $[k]$ ($[f(x)]$ denotes the polynomial gotten by replacing the coefficients by their residue classes in $[k]$), then we contend that $f(x)$ must be reducible in $k[x]$. For, let K be the splitting field of $f(x)$ over k and $f(x) = (x - A_1)(x - A_2) \cdots (x - A_n)$ in $K[x]$. Since $[x - A_1]\,[x - A_2] \cdots [x - A_n] = [g(x)]\,[h(x)]$, there exist A_i, A_j with $[x - A_i] \mid [g(x)]$, $[x - A_j] \mid [h(x)]$, i.e.,

$$| g(x) - (x - A_i)l(x) | < 1 \quad \text{for some} \quad l(x) \;\varepsilon\; K[x],$$

$$| h(x) - (x - A_j)m(x) | < 1 \quad \text{for some} \quad m(x) \;\varepsilon\; K[x].$$

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