NON-COMMUTATIVE SEMI-LOCAL AND LOCAL RINGS

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Introduction. In this paper we study the basic properties of a class of rings which, in the commutative case, reduce to the well known semi-local and local rings [1], [2], [5]. In the first part we give a statement of all the results on rings with ideal nuclei which we shall assume in this paper. Since the proofs of these results are virtually the same as in the commutative case, no proofs have been included, and the reader is referred to [7] and [9]. Part Two is devoted to semi-local rings, and basic structure theorems for complete semi-local rings are developed by means of a technique due to Kaplansky [4]. Part Three is concerned with local rings, and the results of part Two are therein given a somewhat sharper form. In the interest of completeness, all the relevant theorems on semi-local and local rings are given. Where the proofs are essentially the same as in the commutative case, we have merely indicated the appropriate reference. Some examples are given at the end of the paper. We plan to utilize some of these ideas in a later paper on integral extensions of commutative local rings [3].

1. Topological preliminaries.

1.1 Definition. Let R be a ring and I a proper two-sided ideal in R; we say I is a nucleus for R if $\bigcap_{n=0}^{\infty} I^n = (0)$, where $I^0 = R$. If R is a ring with nucleus I, we may introduce the structure of a topological space into R by taking for a fundamental system of neighborhoods of zero the ideals I^n, $n = 0, 1, 2, \cdots$. We can then define a metric in R as follows: $d(x, y) = e^{-k}$ if $x \equiv y \bmod I^k$, $x \not\equiv y \bmod I^{k+1}$, $d(x, y) = 0$ if $x = y$. Of course, in place of e we could use any $n > 1$; the two metrics thus obtained are equivalent. We refer to the topology thus obtained as the natural topology induced in R by I. R is totally disconnected in this topology and the ring operations are continuous. If $\{x_i\}$, $i = 1, 2, 3, \cdots$, is a sequence of elements in R, convergence and regularity are defined with respect to the natural topology. As usual, R will be said to be complete if every regular sequence in R has a limit in R.

Since the ideals I^n, $n = 0, 1, 2, \cdots$, form a fundamental system of neighborhoods of zero in R, the sets $x + I^n$, $n = 0, 1, 2, \cdots$, form a local base at x for R. Thus, if A is any subset of R and $A^{\blacktriangle}$ its closure, $A^{\blacktriangle} = \bigcap_{n=0}^{\infty} (A + I^n)$. In particular, if A is an ideal of R, A is closed if and only if $A = \bigcap_{n=0}^{\infty} (A + I^n)$. The following result will be used throughout the paper.

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