A NOTE ON THE BESSEL POLYNOMIALS

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1. Krall and Frink [10] defined the polynomial

$$y_n(x) = \sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!r!} \left(\frac{x}{2}\right)^r, \tag{1.1}$$

which satisfies the differential equation

$$x^2y'' + (2x+2)y' - n(n+1)y = 0, \tag{1.2}$$

and proved a number of properties of $y_n(x)$ as well as of a generalized polynomial $y_n(x, a, b)$ which reduces to $y_n(x)$ for $a = b = 2$. Burchnall [3] and Grosswald [9] found additional properties of these polynomials; see also [13], [14]. Burchnall defined

$$\theta_n(x) = x^n y_n\left(\frac{1}{x}\right). \tag{1.3}$$

It is also convenient to let

$$y_{-n}(x) = y_{n-1}(x), \qquad \theta_{-n}(x) = x^{1-2n}\theta_{n-1}(x). \tag{1.4}$$

It follows from (1.4) that, for example, the recurrence formula satisfied by y_n and θ_n hold for all integral n.

In the present note we derive some more formulas satisfied by the polynomials. Some of the results are simpler when stated in terms of the polynomial $f_n(x)$ defined by

$$f_n(x) = x^n y_{n-1}\left(\frac{1}{x}\right) = x\theta_{n-1}(x); \tag{1.5}$$

it is convenient to complete the definition by means of

$$f_{-n}(x) = x^{-1-2n}f_{n+1}(x), \qquad f_0(x) = 1. \tag{1.6}$$

2. Since [3; 64]

$$\theta_{n+1} = (2n+1)\theta_n + x^2\theta_{n-1}, \tag{2.1}$$

$$\theta_n' = \theta_n - x\theta_{n-1}, \tag{2.2}$$

it follows at once that

$$f_{n+1} = (2n-1)f_n + x^2 f_{n-1}, \tag{2.3}$$

$$f_n' = f_n - xf_{n-1}. \tag{2.4}$$

These formulas hold for all integral n.

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