A METHOD FOR APPROXIMATING THE ZEROS OF ANALYTIC FUNCTIONS

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Introduction. In this paper a zero a of an analytic function $f(z)$ is determined as the limit of a function $F_n(z)$:

$$a = \lim F_n(z), \quad \text{as} \quad n \to \infty, \tag{1}$$

where z may be any number, which is closer to the zero, a, than to any other zero of $f(z)$. The function $F_n(z)$ has the form $z - f(z)\, Q_{n-1}/Q_n$ or $z - f(z)/Q_n^{1/n}$ as is given in (10) and (11). The function $Q_n = Q_n(z)$ is given in (5). It is a homogeneous function of $f(z)$ and its $n - 1$ first derivatives and depends also on another appropriately chosen function, $\phi(z) = \phi(z; u, v, \cdots, t)$.

The algorithm,

$$z_{k+1} = F_n(z_k), \tag{2}$$

can be used for successive approximations to a zero of $f(z)$. It is only a first order iteration formula in general case. However, the convergence to the zero is at least of order n or $n + 1$ in particular cases, depending on the multiplicity of the zero to be approximated and the choice of $\phi(z)$ and its parameters.

The general method. Let $f(z)$ be a function analytic within and upon the circle C, having zeros a_i in the defined area. Let $\phi(z) = \phi(z; u, v, \cdots, t)$ be another such function, which may depend on a number of parameters. If $\phi(z)$ has a zero in common with that zero of $f(z)$ which we intend to evaluate, it has this zero with multiplicity less than $f(z)$ does. Otherwise, $\phi(z)$ is an arbitrary function.

We may write

$$\phi(z)/f(z) = \Sigma(z - a_i)^{-m_i}\, R_{1i}(z) + \psi_1(z), \tag{3}$$

where the sum has a finite number of terms and is taken over all the zeros a_i of $f(z)/\phi(z)$ within and upon the circle C, their multiplicities being m_i ; $R_{1i}(z)$ is a polynomial in $z - a_i$ of degree less than m_i with constant term distinct from zero; and $\psi_1(z)$ is analytic within and upon C.

On differentiating both members of (3) $n - 1$ times and dividing by $(n - 1)!$ $(-1)^{n-1}$ we get

$$Q_n/[f(z)]^n = \Sigma(z - a_i)^{-n-m_i+1}\, R_{ni}(z) + \psi_n(z), \tag{4}$$

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