# LOCALLY COMPACT TRANSFORMATION GROUPS

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Let $X$ be a locally compact Hausdorff space, $T$ a group of homeomorphisms of $X$, and $\pi$ the mapping of $X \times T$ into $X$ such that $\pi(x, t) = xt$ for all $x \in X$ and $t \in T$. The purpose of this paper is to prove that if $T$ is provided with a locally compact Hausdorff topology such that $\pi$ is unilaterally continuous and the maps $t \rightarrow st$ and $t \rightarrow ts$ of $T$ into $T$ are continuous for all $s \in T$, then $\pi$ is continuous. Using this result, it is shown that if $X$ is a locally compact Hausdorff space with a group structure such that the maps $x \rightarrow xy$ and $x \rightarrow yx$ of $X$ into $X$ are continuous for all $y \in X$, then $X$ is a topological group.

In the sequel the following notation will be used. Let $X$ and $Y$ be topological spaces. Then $C(X, Y)$ will denote the set of continuous maps of $X$ into $Y$. The symbols $C_p(X, Y)$ and $C_c(X, Y)$ will denote the set $C(X, Y)$ provided with the topologies of pointwise convergence and uniform convergence on compact sets respectively. [1] If $\mathfrak{I}$ is a topology on $X$ and $A \subset X$, then $A \wedge \mathfrak{I}$ will denote the topology induced on $A$ by $\mathfrak{I}$. Let $T \subset C(X, Y)$, then $\pi : X \times T \rightarrow Y$ will denote the map such that $\pi(x, t) = xt$ for all $x \in X$ and $t \in T$.

Let $T \subset C(X, X)$ and let $\mathcal{S}$ be a topology on $T$. Then $(T, \mathcal{S})$ will be called admissible if the following conditions are satisfied.

(i) $T^2 \subset T$.

(ii) If $t \in T$, then $t$ is onto.

(iii) The topology $\mathcal{S}$ is locally compact and $\mathcal{S} \supset T \wedge C_p(X, X)$.

(iv) The maps $t \rightarrow ts$ and $t \rightarrow st$ of $T$ into $T$ are continuous for all $s \in T$.

1. In this section it is assumed that $X$ is a compact metric space, $(T, \mathcal{S})$ admissible, and $G$ a group of homeomorphisms of $X$ such that $G \subset T$ and cls $G = T$.

Lemma 1. *$T$ is first countable.*

*Proof.* Let $t \in T$ and let $V$ be a compact neighborhood of $t$. Since $X$ is compact metric, it is separable, and so there exists a countable subset $E$ of $X$ with cls $E = X$. Now $V \wedge \mathcal{S} \supset V \wedge C_p(X, X) \supset V \wedge C_p(E, X)$, and since $C_p(E, X)$ is Hausdorff and $V \wedge \mathcal{S}$ is compact, these three topologies coincide. However, $C_p(E, X)$ is metrizable and thus so is $V \wedge \mathcal{S}$. The proof is completed.

Lemma 2. *The set $A$ consisting of the one-one elements of $T$ is residual.*

*Proof.* Let $(V_n/n = 1, \cdots)$ be a neighborhood base of the identity element $e$ of $T$ consisting of open sets. Set $A_n = [t/ts \in V_n$ for some $s \in T]$. Then $A_n$ is open since $(T, \mathcal{S})$ is admissible. Moreover $G \subset A_n$, because $G$ is a group. Then for each $n$, $A_n$ is an everywhere dense open set.

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