BOUNDEDNESS AND CONVERGENCE OF SOLUTIONS OF

$x'' + cx' + g(x) = e(t)$

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1. **Introduction.** In the present paper we establish a boundedness theorem and a convergence theorem for solutions of the differential equation

$$x'' + cx' + g(x) = e(t) \qquad (' = d/dt). \tag{1.1}$$

For the boundedness theorem we consider the more general equation

$$x'' + f(x)x' + g(x) = e(t) \tag{1.2}$$

under the assumption that $f(x)$ is everywhere greater than or equal to some positive constant. Theorems of this nature have been established by Cartwright [2], Cartwright and Littlewood [3], Levinson [5], and Reuter [12], [13]. These authors all permit $f(x)$ to be negative for small values of x. We are not concerned here with negative damping. By restricting $f(x)$ to positive values we are able to obtain simple explicit bounds for the solutions of (1.2).

We establish a convergence theorem for solutions of (1.1) with c a positive constant. Levinson [6], Cartwright [2], and Cartwright and Littlewood [3] have established convergence theorems for solutions of (1.2). They all require $f(x)$ to be nonnegative. In [6] $g(x)$ is linear. In [2] and [3] conditions are imposed on $g''(x)$. In our result $f(x)$ is required to be constant, but no condition is imposed on $g''(x)$.

A theorem on the growth of solutions of the linear differential equation

$$y'' + a(t)y = 0 \tag{1.3}$$

is given. Exponential bounds for solutions of (1.3) are given in terms of bounds for $| a(t)|$. The technique is due to Liapounoff [8], [9].

2. **A boundedness theorem.**

Theorem 1. *In the differential equation*

$$x'' + f(x)x' + g(x) = e(t) \tag{2.1}$$

let $f(x)$, $g(x)$, and $e(t)$ be such as to guarantee existence of solutions. Let positive constants c, b, and E exist such that for all values of their variables

$$f(x) \geq c, \qquad g'(x) \geq b, \qquad | e(t) | \leq E. \tag{2.2}$$

Received June 8, 1956. The research for this paper was sponsored in part by the Office of Ordnance Research, U. S. Army, Contract No. DA-11-022-ORD-1869. This paper was written while the author was a Visiting Fellow at the Massachusetts Institute of Technology.