SELF-ADJOINT, NON-OSCILLATORY SYSTEMS OF ORDINARY, SECOND ORDER, LINEAR DIFFERENTIAL EQUATIONS

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Introduction. A theorem of A. Kneser [5; 183–191] implies that if the differential equation

$$x'' + f(t)x = 0 \tag{i}$$

is non-oscillatory on $0 \leq t < \infty$ by virtue of the condition $f(t) \leq 0$, then there exists a *principal* solution $x = y(t) \not\equiv 0$ which is unique up to constant factors and has the property that if $x = x(t)$ is any *non-principal* solution, that is, any solution linearly independent of $y(t)$, then

$$y(t)/x(t) \to 0 \quad \text{as} \quad t \to \infty; \tag{ii}$$

in fact,

$$| y(t) | \leq \text{Constant} \quad \text{and} \quad | x(t) | \to \infty \quad \text{as} \quad t \to \infty. \tag{iii}$$

Furthermore,

$$\int^{\infty} dt/y^2(t) = \infty \quad \text{and} \quad \int^{\infty} dt/x^2(t) < \infty. \tag{iv}$$

It turns out that, whether or not $f(t) \leq 0$, the assertions concerning (ii) and (iv), but not (iii), are true whenever (1) is non-oscillatory. This fact was proved in [6; 254–256]; cf. also [1; 703] and [3; 480–483], where the existence of the proof in [6] was unfortunately overlooked. (As to the assertion (iii) when the condition $f(t) \leq 0$ is dropped, cf. [2; 633–645].) The existence and properties of a principal solution are easily transferred from (i) to an arbitrary non-oscillatory, self-adjoint equation

$$(p(t)x')' + f(t)x = 0, \qquad p(t) > 0. \tag{v}$$

The object of this paper is to extend the notion of a principal solution $x = y(t)$ belonging to a non-oscillatory, scalar equation (i) to that of a principal solution belonging to a non-oscillatory (cf. [8; 313]), self-adjoint system of equations, say

$$x'' + F(t)x = 0, \tag{vi}$$

where x is a vector and $F(t)$ is an Hermitian matrix.

The results are known [10] in the analogue of the case treated by Kneser, that is, when $F = F^*$ is real and non-positive definite or, more generally, if

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