THE ORDER OF THE ZETA FUNCTION IN THE CRITICAL STRIP

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In [2] use is made of the following

Theorem A (van der Corput). *If* $l \geq 3$, $2L = 2^l$, $\sigma = 1 - l/(2L - 2)$, $\zeta(s) = O(t^{1/(2L-2)}) \log t)$.

This theorem has been proved only with the additional hypothesis that l is an integer, but in the statement of the theorem in [5], this hypothesis is not explicitly stated. Professor L. Schoenfeld has kindly called my attention to the necessity of justifying its use in [2] for non-integral l. In order to fill the gap, we give here a proof of the following Theorem 1; the proof uses a result of Min [3], but is, otherwise, essentially an elaboration of a result of [1].

Theorem 1. *For arbitrary* $l \geq 3$, *let* $2L = 2^l$, $\sigma = 1 - l/(2L - 2)$ *and set* $\lambda = [l] - 1$, $g = l - [l]$. *Then, for every* $\epsilon > 0$, $\zeta(s) = O(t^{\mu+\epsilon})$, *with* $\mu = \alpha/(2L - 2)$ *and* (i) $\alpha \leq 1 + (1 + g - 2^g)/(\lambda + 2^{-\lambda}) \leq 1.0276$ *for* $l \geq 4$; (ii) $\alpha \leq \{59l - 7(2L - 2)\}/138 \leq 1.0254$ *for* $3 \leq l \leq 4$; (ii) *is equivalent to* (iii) $\mu(\sigma) \leq (52 - 59\sigma)/138$ *for* $1/2 \leq \sigma \leq 5/7$.

Using (iii) instead of [5, Theorem 5.14] in [2], with $\sigma = \log 2/\log 3$, one obtains $\mu \leq .10706$ and $c \leq \log 2/\log 3 + 2\,\mu \leq .8450$, instead of .8385.

Proof of Theorem 1. With the integers $l_2 = l_1 + 1 = \lambda + 2 \geq 4$, define $2L_j = 2^{l_j}$, $\sigma_j = 1 - l_j/(2L_j - 2)$ $(j = 1, 2)$. Then, by Theorem 5.14 in [5], $\zeta(s_j) = O(t^{\mu_j+\epsilon})$, where $s_j = \sigma_j + it_j$ and $\mu_j \leq 1/(2L_j - 2)$. If $l = l_1 + g$, $l_1 \leq l = l_1 + g \leq l_2$, then $\sigma_1 \leq \sigma = 1 - l/(2L - 2) \leq \sigma_2$ and $\sigma = \sigma_1 + k(\sigma_2 - \sigma_1)$ with $0 \leq k = (\sigma - \sigma_1)/(\sigma_2 - \sigma_1) \leq 1$. By the convexity of $\mu = \mu(\sigma)$,

$$\mu(\sigma) \leq \mu(\sigma_1) + k\{\mu(\sigma_2) - \mu(\sigma_1)\} = \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}\,\mu(\sigma_2) + \frac{\sigma - \sigma_2}{\sigma_1 - \sigma_2}\,\mu(\sigma_1).$$

Replacing here σ, σ_j and μ_j by their values, after routine simplifications, one obtains $\mu \leq \{1 + (1 + g - 2^g)/(\lambda + 2^{-\lambda})\}/(2L - 2)$. The function $1 + g - 2^g$ attains its maximum for $2^g = (\log 2)^{-1}$; hence, $1 + g - 2^g \leq 1 - (\log 2)^{-1} \log(e \log 2) \simeq .08604 \cdots$. The denominator $\lambda + 2^{-\lambda} \geq 3 + 2^{-3} = 3.125$, provided that $l \geq 4$ and (i) follows. In case $l_1 = 3$, $\sigma_1 = \frac{1}{2}$, $\sigma_2 = 5/7$, $\sigma = \frac{1}{2} + k(5/7 - \frac{1}{2}) = \frac{1}{2} + 3k/14$ and $k = 14\,(\sigma - \frac{1}{2})/3$. Taking $\mu(\sigma_1) = 15/92$ (see [3]) and $\mu(\sigma_2) = 1/(2L_2 - 2) = 1/14$, and using the convexity of $\mu(\sigma)$, we obtain $\mu(\sigma) \leq 15/92 + k(1/14 - 15/92) = 15/92 - (59/7.92)\,(14\,(\sigma - \frac{1}{2})/3) = (52 - 59\sigma)/138$, proving (iii). Replacing σ by $1 - l/(2L - 2)$, $\mu \leq 52/138 - (59/138)\,(1 - l/(2L - 2)) = \alpha/(2L - 2)$, with $\alpha \leq (-7(2L - 2) + 59l)/138$. The numerator is maximum for $2^l = 59/7\,(\log 2)$, i.e. for $l \simeq 3.604 \cdots$; hence, $\alpha \leq 141.52/138 \simeq 1.0254 \cdots$, finishing the proof.

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