

GROUPS OF MOTIONS TRANSITIVE ON SETS OF GEODESICS

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1. Introduction. If a space possesses for any two line elements a motion taking the first into the second, then it evidently also possesses a motion taking one of two given geodesics into the other. The present note establishes the *converse* for compact spaces. Since H. C. Wang determined all compact spaces of the former type in [5], those of the latter type are also known (see Theorem 1). Except for two and, under certain differentiability hypotheses, for three dimensions (Theorem 3), the corresponding problem is open for unbounded spaces.

We also exhibit an interesting phenomenon that can occur in non-compact spaces: An n -dimensional space ($n \geq 2$) may possess a transitive (total) group of motions which acts also transitively on the geodesics with the exception of those in a single $(n - 1)$ -parameter family.

Regarding the space we assume that it is a G -space; see [1; 37]; the definitions and results of [1] will be freely used. At first sight the statement that a motion Φ takes the geodesic g into the geodesic h seems ambiguous: it may mean either that representations (see [1; 32]) $x(\tau)$ of g and $y(\tau)$ of h exist such that $x(\tau)\Phi = y(\tau)$. $-\infty < \tau < \infty$; or that $G\Phi = H$, if G and H denote the pointsets carrying g and h . However, the two interpretations are equivalent [1; 39].

2. Compact Spaces. Consider a *compact* G -space R of dimension at least 2, and denote by Γ the group of all motions of R . Then Γ is compact and a Lie group, [1, (4.15) and (52.4)]. We assume that Γ *contains for any two geodesics in R a motion taking the first into the second*. Then the dimension of Γ is positive, so that Γ has one-parameter subgroups $\Phi_u, \Phi_{u_1+u_2} = \Phi_{u_1}\Phi_{u_2}$. A one-parameter subgroup possesses an orbit which is a geodesic g , i.e., for a suitable representation $x(\tau)$ of g and a suitable $\alpha > 0$ we have $x(\tau) = x(0)\Phi_{\alpha\tau}$, [1, (52.2)]. Changing the parameter u we obtain $x(\tau) = x(0)\Phi_\tau$.

Let h be any second geodesic and Ψ a motion taking g into h , so that for a suitable representation $y(\tau)$ of h the relation $x(\tau)\Psi = y(\tau)$ holds for all τ . Then

$$y(0)\Psi^{-1}\Phi_\tau\Psi = x(0)\Phi_\tau\Psi = x(\tau)\Psi = y(\tau).$$

Hence each geodesic is the orbit of a one-parameter group of motions.

Now let two pairs of points a, b and a', b' with $ab = a'b' = \delta > 0$ be given. Let $x(\tau)$ and $x'(\tau)$, $0 \leq \tau \leq \delta$, represent segments $T^+(a, b)$ and $T^+(a', b')$, [1; 27]; extend $x(\tau)$ and $x'(\tau)$ to representations of geodesics, [1, (7.9) and (8.4)]. The existence of a motion Ψ taking the first geodesic into the second means that for a suitable $\eta = \pm 1$ and a suitable real β

$$x(\tau)\Psi = y(\eta\tau + \beta) \text{ for all } \tau \text{ and } x(\tau_1)x(\tau_2) = y(\eta\tau_1 + \beta)y(\eta\tau_2 + \beta)$$

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