THE BANACH ALGEBRA l^1 WITH AN APPLICATION TO LINEAR TRANSFORMATIONS

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Let l^1 denote the Banach algebra of all functions $f(z)$ analytic in the unit circle and with absolutely summable Taylor coefficients, the norm being defined by $\| f \| = \sum_{n=0}^{\infty} | a_n |$. In this paper it will be shown that the quotient algebra l^1/I where I is a closed ideal is reflexive if and only if the space is finite dimensional. This result is then used to establish a general theorem concerning linear transformations of a Banach space into itself.

It should be noted that the algebra l^1 differs from many commutative Banach algebras in that no element of the algebra can vanish on an open set of maximal ideals. Thus the results obtained, for example in [2], for algebras satisfying a Ditkin condition are not applicable to l^1.

In what follows there will be frequent occasion to make use of the concept of the polar of a set. If H is a subset of a Branch space E with conjugate space E^*, the polar of H is the set of all f in E^* for which $\sup_{x \varepsilon H} |(x, f)| \leq 1$ and is written H^0. H^0 is circled, convex and weak-star closed. (A set is called circled if it is invariant under multiplication by complex scalars of modulus ≤ 1.) H^{00} is the polar of H^0 and is a subset of the second conjugate of E. It is well known that if H is a subspace of E, it will be reflexive if and only if the canonical mapping of E into the second conjugate space carries H onto H^{00} [1].

The conjugate space for l^1 is the space m consisting of all bounded sequences $X = X(n)$ $n \geq 0$ of complex numbers with norm $\| X \| = \sup_n | X(n)|$, while l^1 itself is the conjugate of c_0, the subspace of m consisting of sequences $X(n)$ for which $\lim_{n=>\infty} X(n) = 0$. In the sequel the letter V will denote the linear transformation defined on l^1 by multiplication by z; i.e. $Vf = g$ where $g(z) = zf(z)$. The adjoint transformation in m is the operator L of left-translation there: $LX = Y$, $Y(n) = X(n + 1)$.

Theorem 1. *If I is a closed ideal in l^1, then l^1/I is reflexive if and only if it is finite-dimensional.*

Proof. It is not difficult to show that the maximal ideal space of the l^1 algebra is the compact unit circle. Corresponding to the closed ideal I there exists a closed set Z of the unit circle which is the hull of I; i.e. the set of all maximal ideals containing I. It will first be shown that Z is a finite set, and from this the finite-dimensionality of l^1/I will be inferred.

If l^1/I is reflexive, then its conjugate space I^0 is a reflexive subspace of m. If Z is infinite, there is a sequence λ_n in Z converging to some λ_0 in Z which is

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