# EXCEPTIONAL VALUES OF ENTIRE FUNCTIONS

By S. K. Singh

1. Let $f(z)$ be an entire function and let $n(r, x)$ be the number of zeros of $f(z)-x$ in $|z| \leq r$; $M(r, f) = \max |f(z)|$, $(|z| = r)$. We call $\rho(r)$ a Lindelöf proximate order for $f(z)$, if

$$\lim_{r\to\infty} \rho(r) = \rho$$

$$\lim_{r\to\infty} (r\rho'(r) \log r) = 0$$

$$\log M(r, f) \leq r^{\rho(r)} \qquad \text{for all } r \geq r_0$$

$$= r^{\rho(r)} \qquad \text{for an infinity of } r.$$

Valiron [4; 87] has shown that if the ratio $n(r, a)/r^{\rho(r)} \to 0$ as $r \to \infty$, then $0 < A \leq \liminf n(r, x)/r^{\rho(r)} \leq \limsup n(r, x)/r^{\rho(r)} \leq B < \infty$ for all $x \neq a$.

If we replace the comparison function $r^{\rho(r)}$ by $r^{\rho}$, the analogy does not hold. Consider for example

$$f(z) = \prod_2^{\infty} \left\{1 + \frac{z}{n(\log n)^2}\right\}.$$

Then $f(z)$ is an entire function of order 1 for which $n(r, x)/r \to 0$ for all values of $x$. As a matter of fact, for functions of minimal type $n(r, x)/r^{\rho} \to 0$ for all $x$, because by Jensen's theorem

$$\log M(2r, f) \geq \int_r^{2r} \frac{n(t, x)}{t}\, dt \geq n(r, x) \log 2$$

so

$$\frac{n(r, x)}{r^{\rho}} \leq \frac{\log M(2r, f)}{(2r)^{\rho}} \cdot \frac{(2r)^{\rho}}{r^{\rho}} \cdot \frac{1}{\log 2} \to 0$$

as $r \to \infty$.

In this paper we take $r^{\rho}L(r)$ as the comparison function to study similar problems, where $L(t)$ is a function of $t$ continuous for all large $t$, such that $L(ct) \sim L(t)$ for every fixed positive $c$. For instance, $L(t)$ can be of the type $l_1tl_2t \cdots l_kt$ or $(l_1tl_2t \cdots l_kt)^{-1}$, where $l_1t = \log t$, $l_2t = \log \log t$, etc. As a particular case $L(t)$ can be a constant, too.

Theorem 1. *If $f(z)$ is an entire function of order $\rho$ $(0 < \rho < \infty)$ for which*

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{r^{\rho}L(r)} = a \qquad (0 \leq a \leq \infty)$$

Received January 20, 1956.