SOME TOTIENT FUNCTIONS

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1. **Introduction.** The Euler ϕ-function, or totient function, has been generalized in a number of ways [4]. The most important such extension is the Jordan function $J_k(r)$ defined, for positive integers k and r, to be the number of ordered sets of k elements from a complete residue system (mod r) such that the greatest common divisor of each set is prime to r [7; 95-97], [4; 147]. A second generalization is von Sterneck's function $H_k(r)$ defined by ([9], [4; 151])

$$H_k(r) = \sum_{r=[d_1, \cdots, d_k]} \phi(d_1) \cdots \phi(d_k), \tag{1.1}$$

where the summation ranges over all ordered sets of k positive integers d_1 , $\cdots$, d_k with least common multiple equal to r. It is clear that $J_1(r) = H_1(r) = \phi(r)$. In fact, $J_k(r)$ and $H_k(r)$ are equivalent [9], and

$$J_k(r) = H_k(r) = r^k \sum_{d \mid r} \frac{\mu(d)}{d^k}, \tag{1.2}$$

where $\mu(d)$ denotes the familiar Möbius function. The evaluation in (1.2) is sometimes used as an alternative definition of the Jordan function.

Suppose now that a and b are integers, not both zero. We define (a, b_k) to be the largest k-th power divisor common to a and b, and in case $(a, b)_k = 1$, we say that a and b are *relatively k-prime*. Further, we shall refer to the subset N of a complete residue system M (mod r^k), consisting of all elements of M that are relatively k-prime to r^k, as a *k-reduced residue system* (mod r). If, in particular, M consists of the numbers a, $0 \leq a < r^k$, then M will be called a *minimal* residue system (mod r^k) and the corresponding subset N, a minimal, k-reduced residue system (mod r).

The number of elements of a k-reduced residue system is denoted by $\phi_k(r)$; in particular, $\phi_1(r) = \phi(r)$. The function $\phi_k(r)$ was introduced in [1] under the name of the Jordan function, but the equivalence of $J_k(r)$ and $\phi_k(r)$ was not actually proved. The totient $\phi_k(r)$ arises naturally as the case $n = 0$ of the author's trigonometric sum $c_k(n, r)$, defined in [1, §1]. The characteristic properties of $\phi_k(r)$ follow as special cases of properties of $c_k(n, r)$ proved in [1, §2]. For completeness, we indicate in §2 several independent proofs of these properties, listed as Theorems 1 through 4. The equivalence of the three functions $J_k(r)$, $H_k(r)$ and $\phi_k(r)$ is established in Theorem 5.

In §3 the number of solutions $Q_k(n, r, s)$ in x_i (mod r), y_i (mod r^k) of the congruence,

$$n \equiv a_1 x_1^k y_1 + \cdots + a_s x_s^k y_s \pmod{r^k}, \qquad (a_i , r) = 1, \tag{1.3}$$

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