THE RATIONAL POINTS IN HILBERT SPACE

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1. **Introduction.** In [1], Erdös proved that the set R of rational points of Hilbert space is 1-dimensional. It follows from the general embedding theorem (see, for example, [2], Theorem V 3, p. 60) that R is homeomorphic to a subset of Euclidean 3-space. The object of the present paper is to show (Theorem 2) that there exists a homeomorphism h mapping R into a subset of the Euclidean plane. The author is indebted to Dr. Erdös for calling his attention to this problem. Theorem 3 gives an explicit formula for a homeomorphism of Hilbert space into the Hilbert cube.

2. **Notation.** Hilbert space, H, is the set of all $x = (x_1, x_2, \cdots)$ such that x_i is real and $\Sigma x_i^2 < \infty$. The Hilbert cube, I_ω, is the subset of H consisting of all x such that $| x_i | \leq 1/i$. If $x \in H$ then x_i will denote the i-th coordinate of x. This notation also occurs in the form $(f(x))_i$, where f is a mapping into H. For $x \in H$, $y \in H$, $d(x, y)$ denotes the distance from x to y defined by the extended Euclidean formula. In particular, the distance from x to the origin, $(\Sigma x_i^2)^{\frac{1}{2}}$, is called the norm of x and denoted $\| x \|$.

The function φ given by the equation

$$\varphi(x) = x/(1 + | x |) \qquad | x | < \infty, \tag{1}$$

is a homeomorphism of the space of real numbers onto the open interval $(-1, 1)$. Its inverse is given by the equation

$$\varphi^{-1}(x) = x/(1 - | x |) \qquad | x | < 1. \tag{2}$$

Note that $\varphi(x)$ is rational if and only if x is rational.

3. **Results.** THEOREM 1. *The function g: $H \to I_\omega$, defined by the equations $(g(x))_i = \varphi(x_i)/i$ $(i = 1, 2, \cdots)$ is continuous and one-to-one, and preserves rationality coordinatewise.*

Let C denote the Cantor discontinuum on the closed interval $[0, 1]$ of the x-axis, p the point $(\frac{1}{2}, 1)$, and let D be the union of all closed intervals px, $x \in C$. The set D is a "Cantor fan".

The set of rational points in I_ω is 0-dimensional (see [2], Example II 9, p. 12) and there is a homeomorphism α taking this set into a subset of C.

THEOREM 2. *For $x \in R$, let $h(x)$ be the point in D lying on the interval joining p to $\alpha(g(x))$, and having y-coordinate equal to $\varphi(\| x \|)$. Then h is a homeomorphism.*

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