CONVERSES OF SCHWARZ'S INEQUALITY

By Richard Bellman

1. Introduction. The well-known inequality of Schwarz states that

$$\left(\int_0^1 u(x)v(x)\,dx\right)^2 \leq \left(\int_0^1 u^2\,dx\right)\left(\int_0^1 v^2\,dx\right), \tag{1}$$

for any two functions u and v belonging to $L^2(0, 1)$. Without additional restrictions upon u and v, there is no non-trivial inequality going in the other direction; i.e., one of the form

$$\left(\int_0^1 u(x)v(x)\,dx\right)^2 \geq k\left(\int_0^1 u^2\,dx\right)\left(\int_0^1 v^2(x)\,dx\right), \tag{2}$$

where $k > 0$.

If, however, u and v are restricted to lie within certain function classes, there do exist inequalities of the above form with k a positive constant dependent upon the classes chosen. The first discussion of problems of this variety occurs in the papers of Blaschke and Pick, [1], for the case where $u(x)$ and $v(x)$ are concave. (An earlier paper, Frank and Pick, Mathematische Annalen, vol. 76 (1915), p. 354, should also be consulted).

The purpose of this paper is to present a general method for attacking these problems which is equally applicable to other function classes and to multi-dimensional versions. Although it does not solve any particular problem completely, it reduces each problem to a particular investigation which in some cases can be carried through completely.

We shall begin with the one-dimensional case, demonstrating

Theorem 1. *Let $u(x)$ and $v(x)$ be concave functions of x for $0 \leq x \leq 1$, normalized by the conditions*

$$\begin{aligned} &\text{(a)} \quad \int_0^1 u^2\,dx = 1, \qquad \int_0^1 v^2\,dx = 1, \\ &\text{(b)} \quad u(0) = u(1) = 0, \qquad v(0) = v(1) = 0. \end{aligned} \tag{3}$$

Then

$$\int_0^1 u(x)v(x)\,dx \geq \frac{1}{2}. \tag{4}$$

This result is certainly contained in the paper of Blaschke and Pick cited above, although apparently not explicitly stated in the above form. The minimum is attained for

$$\begin{aligned} u(x) &= x\sqrt{3}, \qquad 0 \leq x < 1, \qquad u(1) = 0, \\ v(x) &= (1 - x)\sqrt{3}, \qquad 0 < x \leq 1, \qquad v(0) = 0. \end{aligned} \tag{5}$$

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