SCHLICHT FUNCTIONS WITH REAL COEFFICIENTS

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1. **Introduction.** We will consider two classes of functions, regular and schlicht, and having real coefficients for $|z| < 1$. Functions of this type are evidently of interest in connection with potential problems involving simply-connected regions that are symmetric with respect to a straight line or circle.

As usual, let S be the class of functions, $w = f(z) = z + a_2z^2 + \cdots$, regular and schlicht for $|z| < 1$. Let S_T be the subclass of $f \varepsilon S$ having all a_k real. If $f \varepsilon S_T$ takes on w_0 at $z = z_0$, then $-f(-z) = g \varepsilon S_T$, and takes on $-w_0$ at $z = -z_0$. Furthermore f takes on $\bar{w}_0$ at $z = \bar{z}_0$. Hence one still obtains the full picture after restricting z_0 to the first quadrant.

The commonly known properties of S_T have been obtained by showing that an extremal problem for the larger class T of typically-real functions was solved by an element of S_T [4; 2]. In this way, e.g. best upper bounds for $|f(z_0)|$ as a function of z_0 were obtained, and it was also shown that

$$m(z_0) = \inf_{f \varepsilon S_T} |f(z_0)| = |z_0/(1 + z_0)^2| \quad \text{if} \quad R(z_0 + 1/z_0) \geq 2.$$

The method fails, however, to yield $m(z)$ when $0 \leq R(z + 1/z) < 2$. In §3 we study some properties of S_T by means of a representation derived from Basilewitsch's differential equation. In particular, certain mappings of S_T into itself which have the property of reducing $|f(z_0)|$ are described. We obtain the rather unexpected result that $m(z)$ assumes a maximum at an interior point of $|z| < 1$.

Let $s(\alpha)$ be the class of functions, $w = h(z) = a_1z + a_2z^2 + \cdots$, regular and schlicht for $|z| < 1$, for which all a_k are real, $0 < a_1 \leq \alpha$, $|h(z)| < 1$. In §4 we obtain the variability region for $h(z_0)$ as h ranges over $s(\alpha)$.

The starting point is the following known representation for a class of slit mappings h, dense in $s(1)$, due to Basilewitsch [1], Tammi [5], and Komatu.

Theorem. *Consider schlicht two-slit domains B, consisting of the unit circle with a Jordan curve (omitting the origin) and its symmetric image with respect to the real w-axis as slits. For every such domain B we can find a real-valued function $\sigma(t)$, $-1 \leq \sigma(t) \leq 1$, continuous on $[0, T]$, which has the following property: Considered as a function of z, the function $h(z, T)$ obtained by solving*

$$(1.1) \qquad \frac{dh(z, t)}{dt} = \frac{h(z, t)^3 - h(z, t)}{1 - 2\sigma(t)h(z, t) + h(z, t)^2}, \qquad 0 \leq t \leq T, \qquad h(z, 0) = z;$$

is an element of $s(1)$, with $h(z, T) = e^{-T}z + \cdots$. Conversely, for every piecewise continuous $\sigma(t)$ on $[0, T]$, $-1 \leq \sigma(t) \leq 1$, the solution of (1.1) is in $s(1)$, and $h(z, T) = e^{-T}z + \cdots$.

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