A CHARACTERIZATION OF FUNCTION-LATTICES

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1. **Introduction.** In Problem #81 [1; 176] Garrett Birkhoff asks for a statement of necessary and sufficient conditions that an abstract lattice be lattice-isomorphic to the lattice of all real-valued, continuous functions on some compact Hausdorff space. The central purpose of this paper is to give such a statement. However, it appears that this can best be done in terms of what will be called real lattice homomorphisms. Hence such homomorphisms are first defined and analysed, and this analysis is then applied to the solution of the Birkhoff problem.

Let X denote a set with points $x, y, \cdots$. Let $C(X, T)$ and $C^*(X, T)$ denote respectively the set of all real-valued functions and of all bounded real-valued functions, which are defined on X and are continuous under a given topology T. With a parallel definition for $C^*(X, T)$, a real lattice homomorphism (r.l.h.) on $C(X, T)$ is understood to be a real-valued function φ defined on $C(X, T)$ and satisfying conditions

(a) $\varphi(f \vee g) = \max [\varphi(f), \varphi(g)]$,
(b) $\varphi(f \wedge g) = \min [\varphi(f), \varphi(g)]$,
(c) $\varphi(r) = r$,

for all elements f, g of $C(X, T)$ and for all constant functions $r(x) \equiv r$. Here $f \vee g$ and $f \wedge g$ denote the usual lattice operations used in real function spaces. For each point x in X, the relation $\varphi(f) = f(x)$ determines a r.l.h. φ on $C(X, T)$. Such a r.l.h. is denoted by φ_x and referred to as a point real lattice homomorphism (p.r.l.h.).

Our purpose makes necessary the study of such r.l.h. More commonly studied, however, are what may be called real ring homomorphisms (r.r.h. and p.r.r.h.). With a parallel definition for $C^*(X, T)$, a real ring homomorphism on $C(X, T)$ is understood to be a real-valued function φ defined on $C(X, T)$ and satisfying conditions

(α) $\varphi(f + g) = \varphi(f) + \varphi(g)$,
(β) $\varphi(f \cdot g) = \varphi(f) \cdot \varphi(g)$,
(γ) $\varphi(r) = r$,

for all elements f, g of $C(X, T)$ and all constant functions $r(x) \equiv r$. Such r.r.h. of $C(X, T)$ stand in 1–1, onto correspondence with the real maximal ring ideals of $C(X, T)$ and, under this form, have been carefully discussed [2; 3]. It is an easy matter to prove that every r.r.h. on $C(X, T)$ is also a r.l.h. The first

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