ON THE OPERATOR EQUATION $BX - XA = Q$

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1. **Introduction.** We will be considering a Banach algebra $\mathfrak{B}$, with elements $A, B, Q, \cdots$, and identity element I. T will be the operator on $\mathfrak{B}$ such that $T(Q) = BQ - QA$ for every Q in $\mathfrak{B}$.

The following two results are to be found in the literature:

Result 1.1. *Let $\mathfrak{B}$ be the algebra of n by n matrices. If the characteristic roots of A are distinct from the characteristic roots of B, then T^{-1} exists and is bounded.*

Proof. See Rutherford [3].

Result 1.2. *Let $\mathfrak{B}$ be the space of bounded operators on a Hilbert space. If there exist real numbers a and b such that $a > b$, $B + B^* \leq b$, and $A + A^* \geq a$, then T^{-1} exists as a bounded operator and has the representation*

$$\text{(i)} \qquad T^{-1}(Q) = -\int_0^\infty e^{Bt} Q e^{-At}\, dt.$$

Proof. See E. Heinz [2]. For an extension of this theorem see Cordes [1].

In this paper we shall develop an operational calculus for T that will shed light on results 1.1 and 1.2.

2. **Definitions and notation.** The *resolvent set* $\rho(A)$ of an element A of a Banach algebra is the set of all complex numbers z such that $(z - A)^{-1}$ is in $\mathfrak{B}$. (We write $(z - A)^{-1}$ for $(zI - A)^{-1}$.) The *spectrum* $\sigma(A)$ of A is the complement of $\rho(A)$ in the complex plane. We agree that ϕ is the empty set. If S_1 and S_2 are subsets of the complex plane, then $S_1 - S_2$ is defined to be the set of all complex numbers z such that for some z_1 in S_1 and z_2 in S_2, $z = z_1 - z_2$.

A set D in the complex plane is a *Cauchy domain* if the following conditions are satisfied:

(i) D is bounded and open;

(ii) D has a finite number of components, the closures of any two of which are disjoint; and

(iii) the boundary of D is composed of a finite positive number of closed rectifiable Jordan curves, no two of which intersect.

We denote the positively oriented boundary of D by $b(D)$.

The following topological theorem is proved in Taylor [4].

Theorem 2.1. *Let F be a closed and G a bounded open subset of the complex plane such that $F \subset G$. Then there exists a Cauchy domain D such that $F \subset D$ and $\overline{D} \subset G$.*

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