MAPPINGS ON INVERSE SETS

By G. T. Whyburn

For topological spaces X and Y a mapping $f(X) = Y$ of X onto Y is *open* or *closed* provided the image of every open or closed set, respectively, in X is open or closed respectively in Y. Similarly, f is *quasi-compact* provided the image of every open (or closed) inverse set in X is open (or closed, respectively) in Y, where an *inverse* set is a set U satisfying the relation $U = f^{-1}f(U)$. All our spaces will be T_1-spaces, i.e., spaces with an open set topology in which the empty set and the whole space are open, all unions and finite intersections of open sets are open, and where for any two distinct points x and y there is an open set containing x but not y.

It is well-known (see *Note* at end of paper) that a $1 - 1$ mapping f having any one of the three properties (open, closed, quasi-compact) just defined is necessarily a homeomorphism (and thus has all three). A natural question arises as to when an arbitrary mapping $f(X) = Y$ will be a homeomorphism on the set on which it is $(1 - 1)$, that is, if E denotes the set of all $x \in X$ satisfying the relation $x = f^{-1}f(x)$, the *kernel* of f as it were, then *when is the mapping* $f \mid E$ *topological*? Now if f is either open or closed, it results at once that this is the case. For an open or a closed mapping is necessarily open or closed respectively on an arbitrary inverse set and thus is open or closed respectively on E so that the result follows. If f is merely assumed quasi-compact, however, the situation is quite different. Quasi-compact mappings are of special interest because the natural mapping of a decomposition of a topological space into disjoint closed sets is quasi-compact, and also the natural decomposition of a space X into point inverses generated by a mapping $f(X) = Y$ has in turn a natural mapping which is topologically equivalent to f if and only if f is quasi-compact.

Example. *There exists a quasi-compact mapping $f(X) = Y$, where X and Y are perfectly separable T_1-spaces, which is not quasi-compact on its kernel set E where it is $1 - 1$. Also f is quasi-compact on $X - E$, and f is not closed on any cross section whatever*; (i.e., *any set in X mapping onto all of Y*).

The space X consists of three disjoint infinite sequences of distinct points (x_i), (y_i) and (z_i) and a point z not in any of the sequences with topology defined so that the sequence (x_i) converges to each separate point y_i, the sequence (y_i) has no limit point and the sequence (z_i) converges to the point z. This can be achieved by defining open sets in X to be all sets of any of the types:

$$x_n\,,\ z_n\,,\ y_n + \sum_m^\infty x_i\,,\ z + \sum_m^\infty z_i\,,$$

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