A FURTHER NOTE ON DEDEKIND SUMS

By L. Carlitz

1. Apostol [1] has proved a transformation formula for

$$G_{m-1}(x) = \sum_{r,s=1}^{\infty} r^{-(m-1)} x^{rs} \qquad (|x| < 1),$$

where m is a fixed even integer > 2. In the formula occur the numbers

$$c_{r,s}(h, k) = \sum_{\mu \pmod k} \overline{B}_r\left(\frac{\mu}{k}\right)\overline{B}_s\left(\frac{h\mu}{k}\right) \qquad (k > 0), \tag{1.1}$$

where r and s are non-negative integers, $\overline{B}_r(x)$ is the Bernoulli function, and the summation is over a complete residue system (mod k). Put

$$f_m(h, k; \tau) = \sum_{s=0}^{m} \binom{m}{s} (k\tau - h)^{m-s-1} c_{m-s,s}(h, k). \tag{1.2}$$

Then the transformation formula can be written in the form

$$G_{m-1}(e^{2\pi i\tau}) = (k\tau - h)^{m-2} G_{m-1}(e^{2\pi i\tau_1}) + \frac{(2\pi i)^{m-1}}{2\cdot m!} f_m(h, k; \tau); \tag{1.3}$$

where

$$\tau_1 = (h'\tau + k')/(k\tau - h), \qquad hh' + kk' + 1 = 0. \tag{1.4}$$

It was proved in [3] that (1.3) implies

$$f_m(h, k; \tau) = (-1)^m \tau^{m-2} f_m\left(-k, h; -\frac{1}{\tau}\right) + \frac{1}{\tau}(B + \tau B)^m, \tag{1.5}$$

where B_m is a Bernoulli number in the even suffix rotation; (1.5) was proved in an elementary way in [2] for all $m \geq 0$.

A generalization of (1.5) can be obtained as follows. Put

$$\tau_2 = (h_1'\tau_1 + k_1')/(k_1\tau_1 - h_1), \qquad h_1h_1' + k_1k_1' + 1 = 0, \tag{1.6}$$

so that

$$G_{m-1}(e^{2\pi i\tau_1}) = (k_1\tau_1 - h_1)^{m-2} G_{m-1}(e^{2\pi i\tau_2}) + \frac{(2\pi i)^{m-1}}{2\cdot m!} f_m(h_1, k_1; \tau_1).$$

Substituting in (1.3) we get

$$\begin{aligned} G_{m-1}(e^{2\pi i\tau}) &= (k\tau - h)^{m-2}(k_1\tau_1 - h_1)^{m-2} G_{m-1}(e^{2\pi i\tau_2}) \\ &\quad + \frac{(2\pi i)^{m-1}}{2\cdot m!}(k\tau - h)^{m-2} f_m(h_1, k_1; \tau_1) + \frac{(2\pi i)^{m-1}}{2\cdot m!} f_m(h, k; \tau). \end{aligned} \tag{1.7}$$

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