DIFFERENTIABLE APPROXIMATIONS TO LIGHT INTERIOR TRANSFORMATIONS

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1. **Introduction.** A function $f\colon A \to B$ will be called *light* if for each point b of B, the inverse image $f^{-1}(b)$ is totally disconnected. That is, $f\colon A \to B$ is light if it is not constant on any continuum. A function $f\colon A \to B$ will be called *interior* if for every open subset U of A, $f(U)$ is an open subset of B.

The concept of light interior transformation was introduced by Stoïlow [1; 2]. He introduced such functions as a generalization of the notion of an analytic function and stressed strongly that lightness and interiority were the fundamental topological properties of analytic functions. Indeed, Stoïlow proved not only that every analytic function is light interior, but also that every light interior function on a 2-dimensional manifold is topologically equivalent to some analytic function.

The work of Stoïlow and Whyburn [3] gives a complete analysis of the local action of light interior transformation on two-manifolds. A theorem which will be employed later [3; 198] states that if f is a light interior function on a 2-dimensional manifold, then for every point z there exists a 2-cell neighborhood of z on which f is topologically equivalent to the exponential mapping $w = z^n$ of the unit disk. A point where n is different from one will be called a *topological critical point.*

Although a light interior transformation is topologically equivalent to an analytic function, it need not be differentiable in any sense. The main results of this paper are contained in the following theorem.

Theorem 3. *Let D be a domain in the plane with closure D^- and let $f : D^- \to R^2$ be continuous on D^- and light interior on D into the plane. Let n be a positive integer. Then for every positive number ϵ, there exists a continuous function $g\colon D^- \to R^2$ such that* (1) *g is light interior on D into R^2,* (2) *$f(z) = g(z)$ for z on the boundary of D,* (3) *g is n times continuously differentiable in x and y,* (4) *g has the same topological critical points as f, and* (5) *$| g(z) - f(z) | < \epsilon$ for all z in D^-.*

In §§2 and 3 it is proved that there exists a triangulation T of D and a light interior transformation b which approximates f and is linear on each simplex of T. In §§4 to 6 it is proved that by a suitable smoothing process b can be approximated by differentiable light interior transformations. The smoothing

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