STRUCTURE OF AW^*-ALGEBRAS. I.

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Introduction. The results of this paper (and the sequel) arose from the question of existence of trace in finite AW^*-algebras [5]. However, they could be described more aptly as a study of subalgebras of algebras of type I [6], [7]. Many of them comprised the author's thesis at the University of Chicago.

Some of the results have been obtained independently by T. Yen [9] in slightly less generality. They are included for the purposes of completeness of this study of subalgebras of algebras of type I.

In this paper we give a necessary and sufficient condition for the existence and uniqueness of a center-valued trace for finite AW^*-algebras. Section 1, 2, and 3 deal with the necessity, and section 4 deals with the sufficiency as well as some consequences of the trace.

Notation will be, mainly, that of [5].

1. Definition. A mapping Tr of a C^*-algebra A with unit 1 into its center Z is called a (central) trace if for $x, y \in A$, α, β complex and $z \in Z$

(1) $\mathrm{Tr}(\alpha x + \beta y) = \alpha \,\mathrm{Tr}(x) + \beta \,\mathrm{Tr}(y)$

(2) $\mathrm{Tr}(zx) = z \,\mathrm{Tr}(x)$

(3) $\mathrm{Tr}(x^*x) \geq 0, = 0 \quad \text{only if} \quad x = 0$

(4) $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$

(5) $\mathrm{Tr}(1) = 1$

Let A be a C^*-algebra with unit 1. In [1] Dixmier associates with every element $a \in A$, a set K_a which is the intersection of the center Z (of A) and the smallest closed, convex set containing all uau^*, u unitary ($uu^* = u^*u = 1$). He proves (in a more general fashion)

(1) If K_a is non-void for all $a \in A$ then a necessary and sufficient condition for the existence of a (unique) central trace on A is that K_a consists of one point and that point is $\mathrm{Tr}(a)$.

(2) If K_a is non-void for all $a \in A$ then A has a central trace if and only if for self-adjoint a, and self-adjoint, central b the existence of λ_i (real numbers) and u_i unitary with $\sum \lambda_i = 0$ and $\sum_{i=1}^{n} \lambda_i u_i a u_i^* \geq b$ implies $b \leq 0$.

We use methods similar to those of Dixmier to prove that K_a is non-void for a class of C^*-algebras slightly more general than AW^*-algebras and that K_a reduces to a point in a certain class of finite AW^*-algebras which includes all finite AW^*-algebras which are AW^*-embedded in algebras of type I (of course, all weakly closed algebras are such).

Received March 22, 1955. This work was supported in part by the Office of Naval Research.