SUMMABILITY OF GENERALIZED BERNSTEIN POLYNOMIALS. I.

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1. **Introduction.** If $f(x)$ is Lebesgue integrable over the closed interval [0, 1] then the generalized Bernstein polynomials here considered are defined as

$$P_n^f(x) = \sum_{\nu=0}^{n} (n + 1)p_{\nu,n}(x) \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} f(t)\, dt, \qquad n = 0, 1, 2, \cdots .$$

where

$$p_{\nu,n}(x) = \binom{n}{\nu} x^\nu (1 - x)^{n-\nu}.$$

It is known that

$$\lim_{n\to\infty} P_n^f(x) = f(x) \tag{1}$$

almost everywhere in [0, 1], more precisely, at every point x where $f(x)$ equals the derivative of its indefinite integral (see [5], [6] or [1]). Another property (see [1]) of the $P_n^f(x)$ is that if $f(x) \in L^p$ [0, 1], $p > 1$, they converge dominatedly in the space L^p and hence also in the mean (of order p) to $f(x)$.

These, among other properties, illustrate the analogy that exists between the polynomials $P_n^f(x)$ and the Fejér means of the partial sums $S_n(x)$ of the Fourier series (c.f. [10]). Extending this parallelism, it seems reasonable to consider the polynomials

$$\mathfrak{P}_n^f(x) \equiv (n + 1)P_n^f(x) - nP_{n-1}^f(x)$$

which would correspond to the partial sums $S_n(x)$. More generally, the question of summability will be introduced. The behaviour of the Lagrange interpolation polynomials (see [8] or [2]), the trigonometric interpolating polynomials (see [11]), and the Legendre series for $f(x)$ (see [4] or [7]) is somewhat similar to that of the $\mathfrak{P}_n^f(x)$.

2. **Asymptotic relation for the $P_n^f(x)$.** We first establish a useful asymptotic expansion for the $P_n^f(x)$.

Theorem 1. *If $f(x)$ is defined in* [0, 1], $|f(x)| \leq M$, *then at every point $x \in$* [0, 1] *where $f''(x)$ exists,*

$$\lim_{n\to\infty} (n + 1)[P_n^f(x) - f(x)] = \tfrac{1}{2}[(1 - 2x)f'(x) + x(1 - x)f''(x)].$$

Proof. In view of Taylor's theorem, as $f''(x)$ exists at x,

$$f(t) = f(x) + (t - x)f'(x) + (t - x)^2[\tfrac{1}{2}f''(x) + \eta(t - x)]$$

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